Math 1

Algebra and Calculus 1

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Winter term 2011/2012

February 20, 2012

Contents

1 Logic, sets and functions

1.1 Logic

Definition 1.1 (Statement)**.** A *statement* is a sentence with the following properties:

1. It states a fact.
2. It is either *true* or *false*.

◁

Example 1.1.

1. "Christmas is in December."
   (is a correct statement and *true*)
2. "Bonn is the capital of Germany."
   (correct statement and *false*)
3. "Are you hungry?"
   (incorrect statement)
4. "I shave every man in my village."
   (correct statement, can be *true* or *false*)
5. "I shave every man in my village who can't shave himself."
   (incorrect statement since impossible to resolve)

◁

Remark: For convenience we write 0 for *false* and 1 for *true*.

Definition 1.2 (Compound statements)**.** Let a and b be statements, then we form the following *compound statements*:

1. *negation*: "not a", written as $\overline{a}$ or $\neg a$
2. *conjunction*: "a and b", written as $a \wedge b$
3. *disjunction*: "a or b", written as $a \vee b$
4. *implication*: "a implies b", written as $a \Rightarrow b$
5. *equivalence*: "a equals b", written as $a \Leftrightarrow b$

and define it by the following truth table:

a	b	$\overline{a}$	$a \wedge b$	$a \vee b$	$a \Rightarrow b$	$a \Leftrightarrow b$
0	0	1	0	0	1	1
0	1	1	0	1	1	0
1	0	0	0	1	0	0
1	1	0	1	1	1	1

◁

Example 1.2. As an example for implication let us look at the following statement: "If it rains the street will be wet." It may be written in the following table:

weather	street	statement is
not raining	dry	true
not raining	wet	true
raining	dry	false
raining	wet	true

It appears unfamiliar that the first two lines support the statement and that only the third line result in *false*. However, the statement does not state that it does not rain or that the street might not get wet by some other means. Hence, the statement remains true except for the combination of rain and a dry street. ◁

Theorem 1.3 (Laws on logic)**.** Let a, b and c be statements, then we have:

1. Associativity:

$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$
$$(a \vee b) \vee c = a \vee (b \vee c)$$

2. Commutativity:

$$a \wedge b = b \wedge a$$
$$a \vee b = b \vee a$$

3. Distributivity over $\wedge$ and $\vee$:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

4. Double negation:

$$\overline{\overline{a}} = a;$$

5. De Morgan's laws:

$$\overline{a \wedge b} = \overline{a} \vee \overline{b}$$
$$\overline{a \vee b} = \overline{a} \wedge \overline{b}$$

6. Implication and indirect proof:

$$a \Rightarrow b = \overline{a \wedge \overline{b}}$$

(See next remark for an explanation.)

$\triangleleft$

Proof. All listed laws may be proved by truth tables. Exemplary we prove the first law of De Morgan:

a	b	$\overline{a}$	$\overline{b}$	$a \wedge b$	$\overline{a \wedge b}$	$\overline{a} \vee \overline{b}$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

The last two columns are equal, hence, the first law of De Morgan holds. $\square$

Remark: The *implication* is often used for theorems: "If statement a holds then statement b holds too." For a false statement a the implication is true for any statement b. Hence, to prove such theorems we focus an a true statement a and look at statement b.

A common way to prove a theorem based on an implication is to perform an *indirect proof*. I.e. we assume that statement a is true and statement b is false. If we then find out that this assumption leads to a contradiction then the original theorem is true.

a	b	$a \Rightarrow b$	$a \wedge \overline{b}$	$\overline{a \wedge \overline{b}}$
0	0	1	0	1
0	1	1	0	1
1	0	0	1	0
1	1	1	0	1

Example 1.3. Indirect proof.

Theorem: $p \in Q \Rightarrow p^2 \neq 2$, i.e. if p is a rational number then p^2 can not be 2.

Here statement a is $p \in \mathbb{Q}$ and statement b is $p^2 \neq 2$. Instead of proving that $a \Rightarrow b$ (or $\overline{a \wedge \overline{b}}$) is true we prove that $a \wedge \overline{b}$ is false.

Proof: Suppose there is a rational number p and that $p^2 = 2$. Since for statement b the sign of p plays no role we focus on positive rational numbers only.

p being rational means there are natural numbers $m, n \in \mathbb{N}$ such that $p = \frac{m}{n}$ and we can assume that m and n are not both even. (Otherwise we can cancel all common 2's.) Then we have

$$p^2 = \frac{m^2}{n^2} = 2 \qquad \text{and thus} \qquad m^2 = 2n^2$$

Hence m^2 is even. If m^2 is even then m is also even. (The square of an odd number is again odd.) With m even we can write for $k \in \mathbb{N}$:

$$m^2 = (2k)^2 = 4k^2 = 2n^2 \quad \text{and thus} \quad n^2 = 2k^2$$

Hence n^2 and subsequently n is even. I.e. m and n are both even which is a contradiction to our assumption. Hence, since $(p \in \mathbb{Q}) \wedge (p^2 = 2)$ is false we find $p \in Q \Rightarrow p^2 \neq 2$ is true. $\triangleleft$

Definition 1.4 (Quantifiers)**.** In addition to the above specified logical operators with one or two arguments we have two more, so-called *quantifiers* with a variable number of arguments:
universal quantifier:
$\forall x$: "for all x"
existential quantifier:
$\exists x$: "there is (exists) an x" $\triangleleft$

Example 1.4.

1. $\forall x \in \mathbb{N}$ we have $x + 1 = 1 + x$

   (Commutativity of addition for any natural number with 1.)

2. $\forall x \in \mathbb{N}\ \exists y \in \mathbb{N}$ with the property $y > x$

   (For any natural number there exists a larger one.)

$\triangleleft$

1.2 Sets

Definition 1.5 (Sets)**.** Georg Cantor (1845 - 1918):

> A *set* is "*a collection of certain well defined objects of our perception or thinking.*"

The objects contained in a set are called *elements* of the set M. For Objects x and a set M we write

$$\begin{array}{ll} x \in M & \text{if } x \text{ is an element of } M \\ x \notin M & \text{if } x \text{ is not an element of } M \end{array}$$

The sequential arrangement of the elements does not change the set. Every element appears only once in a set. ◁

Example 1.5.
$\{a, b, c, d\}$ is a set.
$\{a, b, c, c\}$ is not a set.
$\{a, b, c\} = \{a, c, b\} = \{b, a, c\} = \{b, c, a\}$ etc.
◁

Remark: In order to define a set M, there are two different ways:

1. Extensional definition: We itemize all elements of the set.

$$M = \{a_1, a_2, a_3, \ldots\}$$

2. Intensional definition: We define the set by its characteristic properties.

$$M = \{x \mid x \text{ with properties } E_1, E_2, E_3, \ldots\}$$

Example 1.6. Extensional definition of sets:

$M_1 = \{0, 1\}$ (set of boolean values)
$M_2 = \{\clubsuit, \spadesuit, \heartsuit, \diamondsuit\}$ (colours of playing cards)
$M_3 = \{a, b, c, \ldots, z\}$ (lower case alphabet)
$M_4 = \{2, 4, 6, 8, \ldots\}$ (positive even numbers)
$M_5 = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (set of integers)

With these sets we have:

$$\begin{array}{cc} 0 \in M_1, & 2 \notin M_1, \\ \heartsuit \in M_2, & 1 \notin M_2, \\ a \in M_3, & A \notin M_3, \\ 4 \in M_4, & 5 \notin M_4, \\ 1 \in M_5, & 1.5 \notin M_5. \end{array}$$

◁

Example 1.7. Intensional definition of sets:

$M_1 = \{x \in \mathbb{Z} \mid x = 2k, k \in \mathbb{Z}\}$ (even numbers)
$M_2 = \{x \in \mathbb{N} \mid x = k^2, k \in \mathbb{N}\}$ (square numb.)
$M_3 = \{x \in \mathbb{R} \mid x^2 - 4x + 3 = 0\}$
$M_4 = \{x \in \mathbb{Q} \mid |x| \leq 2\}$

With these sets we have:

$$\begin{array}{cc} -4 \in M_1, & 3 \notin M_1, \\ 9 \in M_2, & -9 \notin M_2, \\ 1 \in M_3, & 2 \notin M_3, \\ 2 \in M_4, & \sqrt{2} \notin M_4. \end{array}$$

◁

Remark: The term $x \in M$ has a logical result which is either true or false and we have:

$$x \in M = \overline{x \notin M} \qquad \text{and} \qquad x \notin M = \overline{x \in M}$$

I.e. an element x either belongs to a set M or not — there is nothing intermediate.

Definition 1.6 (Subsets, equal- and empty sets)**.** Let A and B be sets.

- We say A is a *subset* of B if every element of A is also an element of B:

$$A \subseteq B \quad \Leftrightarrow \quad \forall a : a \in A \Rightarrow a \in B$$

- We say A *equals* B if every element of A is also an element of B and vice versa:

$$\begin{array}{rcl} A = B & \Leftrightarrow & \forall a : a \in A \Leftrightarrow a \in B \\ \text{or:} \quad A = B & \Leftrightarrow & A \subseteq B \wedge B \subseteq A \end{array}$$

- We say A is a *proper subset* if A is a subset of B but A does not equal B:

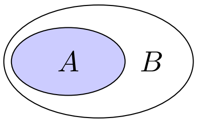
$$A \subset B \quad \Leftrightarrow \quad A \subseteq B \wedge A \neq B$$

  (Obviously, $A \neq B$ means $\overline{A = B}$)

- We say A is an *empty set* if it contains no elements:

$$\emptyset = \{\ \}$$

◁



$A \subset B$

Example 1.8. With $A = \{1, 2, 3\}$, $B = \{1, 2, 3\}$ and $C = \{1, 2, 3, 4\}$ we have

$$A \subseteq B, \qquad A = B, \qquad A \subset C$$

An empty set:

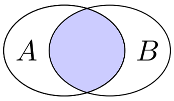
$$\{x \in \mathbb{R} \mid x^2 + 1 = 0\} = \emptyset$$

◁

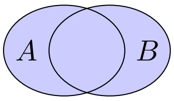
Definition 1.7 (Union, intersection, complement)**.** With A and B being sets and U the set of all possible elements we define the following operators:

$$\begin{aligned}
A \cup B &= \{x \mid x \in A \vee x \in B\} && \text{(union)}\\
A \cap B &= \{x \mid x \in A \wedge x \in B\} && \text{(intersection)}\\
A \setminus B &= \{x \in A \mid x \notin B\} && \text{(complement)}\\
A^C = \overline{A} &= \{x \in U \mid x \notin A\} && \text{(absolute compl.)}
\end{aligned}$$

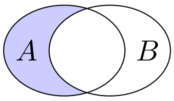
◁



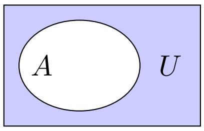
$A \cap B$



$A \cup B$



$A \setminus B$



$\overline{A}$

Remark: The absolute complement equals the complement of U to A:

$$\overline{A} = U \setminus A$$

Theorem 1.8 (Laws on sets)**.** Let A, B and C be sets, then we have:

1. Associativity:
$$\begin{aligned}(A \cap B) \cap C &= A \cap (B \cap C)\\ (A \cup B) \cup C &= A \cup (B \cup C)\end{aligned}$$
2. Commutativity:
$$\begin{aligned}A \cap B &= B \cap A\\ A \cup B &= B \cup A\end{aligned}$$
3. Distributivity over $\cap$ and $\cup$:
$$\begin{aligned}A \cap (B \cup C) &= (A \cap B) \cup (A \cap C)\\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C)\end{aligned}$$
4. Double absolute complement:
$$\overline{\overline{A}} = A$$
5. De Morgan's laws:
$$\begin{aligned}\overline{A \cap B} &= \overline{A} \cup \overline{B}\\ \overline{A \cup B} &= \overline{A} \cap \overline{B}\end{aligned}$$

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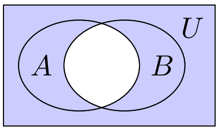
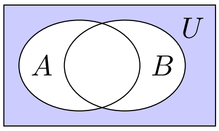


$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$



$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$

Proof. The proof is performed by the elements of the sets. Exemplary we prove the first law of De Morgan for any $x \in U$.

$$\begin{aligned}
x \in (\overline{A \cap B}) &\Leftrightarrow x \notin (A \cap B) \Leftrightarrow \overline{x \notin (A \cap B)}\\
&\Leftrightarrow \overline{x \in A \wedge x \in B}\\
&\Leftrightarrow \overline{x \in A} \vee \overline{x \in B}\\
&\Leftrightarrow x \notin A \vee x \notin B\\
&\Leftrightarrow x \in \overline{A} \vee x \in \overline{B}\\
&\Leftrightarrow x \in (\overline{A} \cup \overline{B})
\end{aligned}$$

□

1.3 Relations

Definition 1.9 (Cartesian product)**.**

Let $M_1, M_2, \ldots, M_n$ be sets, then the set of all n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in M_i$ for $i = 1, \ldots, n$ is called the *Cartesian Product* of $M_1, M_2, \ldots, M_n$ and is denoted by $M_1 \times M_2 \times \ldots \times M_n$, i.e.

$$\begin{aligned}&M_1 \times M_2 \times \ldots \times M_n\\ &\quad = \{(a_1, a_2, \ldots, a_n) \mid a_i \in M_i, i = 1, \ldots, n\}\end{aligned}$$

◁

Remark: The tuple-elements of a Cartesian product are ordered, e.g. for an Cartesian product of two sets we have:

$$(a_1, a_2) \neq (a_2, a_1)$$

I.e. for two sets A and B we have

$$A \times B \neq B \times A$$

unless A and B are equal.

Example 1.9. Each card of the 52 cards of a standard playing card set (excluding jokers) has a rank (2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king or ace) and a colour (♣, ♠, ♡, ♢).

With

$$R = \{2, 3, 4, 5, 6, 7, 8, 9, 10, \mathrm{J}, \mathrm{Q}, \mathrm{K}, \mathrm{A}\}$$

and

$$C = \{\clubsuit, \spadesuit, \heartsuit, \diamondsuit\}$$

we may look at the playing cards P as the Cartesian product of R and C:

$$\begin{aligned} P = R \times C = &\{(2, \clubsuit), (2, \spadesuit), (2, \heartsuit), (2, \diamondsuit), \\ &(3, \clubsuit), (3, \spadesuit), \ldots, (A, \heartsuit), (A, \diamondsuit)\} \end{aligned}$$

◁

Example 1.10. All possible positions in a three dimensional space $\mathbb{R}^3$ may be described by the Cartesian product of three sets of real numbers $\mathbb{R}$:

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x_1, x_2, x_3) \mid x_i \in \mathbb{R}\}$$ ◁

Definition 1.10 (Binary relation)**.** With A and B being sets we call

$$R \subseteq A \times B$$

a *binary relation*. ◁

As the name *binary relation* suggests it describes the relation between two sets.

Example 1.11. Let x and y be natural numbers. The term $x \geq y$ may be looked at as a binary relation R which is a subset of $\mathbb{N} \times \mathbb{N}$:

$$\begin{aligned} R &= \{(x, y) \mid x \geq y, x \in \mathbb{N}, y \in \mathbb{N}\} \\ &= \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3), \ldots\} \\ &\subset \mathbb{N} \times \mathbb{N} \end{aligned}$$

and we get

$$(x, y) \in R \quad \Leftrightarrow \quad x \geq y$$

x ⟶

y ↓

	1	2	3	4	5	6	7
1	✓	✓	✓	✓	✓	✓	✓
2		✓	✓	✓	✓	✓	✓
3			✓	✓	✓	✓	✓
4				✓	✓	✓	✓
5					✓	✓	✓
6						✓	✓
7							✓

◁

Definition 1.11 (Function)**.** Let A, B be sets, $R \subseteq A \times B$ be a binary relation and let R be such that

1. for all $x \in A$ there is a $y \in B$ such that $(x, y) \in R$
2. if $(x, y_1), (x, y_2) \in R$ then $y_1 = y_2$.

If 1 and 2 are satisfied, we can interpret R as a *function* $f : A \to B$.

We call A the *domain* and B the *codomain*. Instead of writing $(x, y) \in R$ we write $x \mapsto f(x)$. ◁

Example 1.12.

1. Square of natural numbers:

$$f : \begin{cases} \mathbb{N} \to \mathbb{N} \\ x \mapsto x^2 \end{cases}$$

2. Cosine of real numbers:

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ \varphi \mapsto \cos \varphi \end{cases}$$

3. Natural logarithm:

$$f : \begin{cases} \mathbb{R}_{>0} \to \mathbb{R} \\ x \mapsto \ln x \end{cases}$$

◁

Definition 1.12 (Identical and constant functions)**.**

1. Let M be a set and f a function. We say f is an *identity function* id_M if

$$f : \begin{cases} M \to M \\ x \mapsto x \end{cases}$$

2. Let X and Y be sets, $y_0 \in Y$ and f be a function. We say f is a *constant function* if

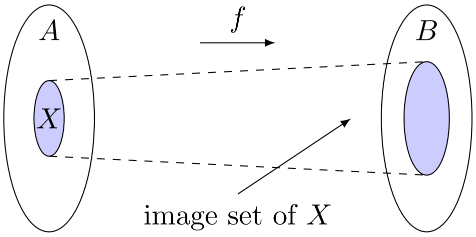
$$f : \begin{cases} X \to Y \\ x \mapsto y_0 \end{cases}$$

◁

Definition 1.13 (Image and inverse image set)**.** Let A and B be sets and $f : A \to B$ a function. For the subsets $X \subseteq A$ and $Y \subseteq B$ we define:

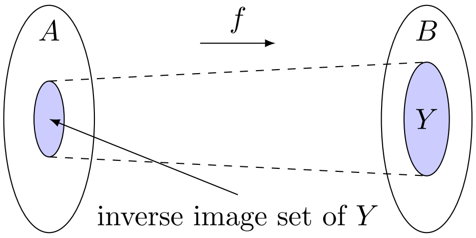
1. *image set* of X:

$$f(X) = \{f(x) \mid x \in X\} \subseteq B$$



2. *inverse image set* of Y:

$$f^{-1}(Y) = \{x \in A \mid f(x) \in Y\} \subseteq A$$



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Remark: By this definition the function f is used in two different ways:

1. The function f maps an element x of its domain to an element y of its range.

$$x \mapsto y = f(x)$$

2. At the same time the function f maps the domain X to the image set Y:

$$X \to Y = f(X)$$

Example 1.13. Let f be a function:

$$f : \left\{ \begin{array}{l} \mathbb{Z} \to \mathbb{Z} \\ x \mapsto x^2 \end{array} \right.$$

then

$$\begin{aligned} f(\{1,2,3\}) &= \{1,4,9\} \\ f^{-1}(\{1,4,9\}) &= \{-3,-2,-1,1,2,3\} \\ f^{-1}(\{2,3\}) &= \{\} = \emptyset \end{aligned}$$

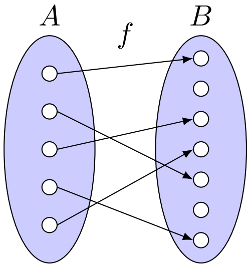
◁

Definition 1.14 (Injective, surjective and bijective functions)**.** Let A and B be sets and f be a function:

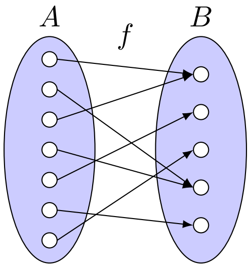
$$f : \left\{ \begin{array}{l} A \to B \\ x \mapsto f(x) \end{array} \right.$$

1. f is called *injective* (one-to-one) if

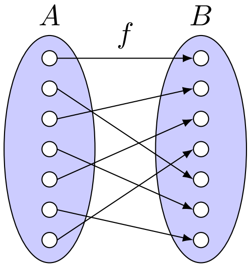
$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$



2. f is called *surjective* (on-to) if for all $y \in B$ there are one or more $x \in A$ such that $f(x) = y$.



3. f is called *bijective* if it is injective and surjective.



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Example 1.14. Let $f : \left\{ \begin{array}{l} \mathbb{N} \to \mathbb{N} \\ x \mapsto 2 \cdot x \end{array} \right.$.

f is *injective*: $2x_1 = 2x_2 \Rightarrow x_1 = x_2$

f is not *surjective*: e.g. 3 is not an image

f is not *bijective*: since f is not surjective. ◁

Example 1.15.

Let $f : \left\{ \begin{array}{l} \mathbb{N} \to \{0,1\} \\ x \text{ even} \mapsto 0, x \text{ odd} \mapsto 1 \end{array} \right.$

f is not *injective*: e.g. $f(1) = f(3) \Rightarrow 1 \neq 3$

f is *surjective*: 0 and 1 are images

f is not *bijective*: since f is not injective. ◁

Example 1.16. Let $f : \left\{ \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{array} \right.$

f is not *injective*: e.g. $f(1) = f(-1) \Rightarrow 1 \neq -1$

f is not *surjective*: e.g. -1 is not an image

f is not *bijective*: since *f* is not injective and not surjective. ◁

Example 1.17. Let $f : \begin{cases} \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \\ x \mapsto x^2 \end{cases}$

f is *injective*: ${x_1}^2 = {x_2}^2 \Rightarrow x_1 = x_2$

f is *surjective*: $f(\mathbb{R}_{\geq 0}) = \mathbb{R}_{\geq 0}$

f is *bijective*: *f* is injective and surjective. ◁

Definition 1.15 (Inverse function)**.** Let X and Y be sets and f be a bijective function $f : X \to Y$. We then call

$$f^{-1} : \begin{cases} Y \to X \\ f(x) \mapsto x \end{cases}$$

an *inverse function*. ◁

Example 1.18. The function

$$f : \begin{cases} \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \\ x \mapsto x^2 \end{cases}$$

has the inverse function

$$f^{-1} : \begin{cases} \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \\ x^2 \mapsto x \end{cases}$$

which is the square root:

$$x = f^{-1}(y) = \sqrt{y}$$

where $y = x^2$. ◁

1.4 Problems

Problem 1.1: Which of the following sets are defined correctly?

1. $M_1 = \{1, 2, 3, 5\}$
2. $M_2 = \{0, 1, 2, a, b, c\}$
3. $M_3 = \{0, 1, \ldots, 9\}$
4. $M_4 = \{0, -1, 1, -2, 2, \ldots\}$
5. $M_5 = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
6. $M_6 = \{\frac{q}{p} \mid q, p \in \mathbb{Z}, p \neq 0\}$
7. $M_7 = \{x \in \mathbb{R} \mid x^2 = 4\}$
8. $M_8 = \{x \in \mathbb{R} \mid x^2 = -1\}$
9. $M_9 = \{x \in \mathbb{Z} \mid x^2 = 2\}$

Problem 1.2: Give an intensional definition of the following sets:

1. $M_1 = \{2, 4, 6, 8, \ldots\}$
2. $M_2 = \{1, 4, 9, 16, \ldots\}$
3. $M_3 = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$
4. $M_4 = \{\frac{2}{1}, \frac{4}{2}, \frac{6}{6}, \frac{8}{24}, \frac{10}{120}, \ldots\}$

Problem 1.3: Give an extensional definition of the following sets:

1. $M_1 = \{x \in \mathbb{N} \mid x = 3n, n \in \mathbb{N}\}$
2. $M_2 = \{x \in \mathbb{N} \mid x = n!, n \in \mathbb{N}\}$
3. $M_3 = \{x \in \mathbb{R} \mid x^2 = 4\}$
4. $M_4 = \{x \in \mathbb{R} \mid x^3 - x^2 - 2x = 0\}$

Problem 1.4: Let A, B and C be statements that are either *true* or *false*. Give the truth table of the following terms:

1. $\overline{A \wedge B}$
2. $\overline{A} \vee \overline{B}$
3. $\overline{A \vee B}$
4. $\overline{A} \wedge \overline{B}$
5. $A \wedge (B \vee C)$
6. $(A \wedge B) \vee (A \wedge C)$
7. $A \vee (B \wedge C)$
8. $(A \vee B) \wedge (A \vee C)$

Problem 1.5: Express the following terms with conjunction and negation only:

1. $A \vee B$
2. $A \Rightarrow B$
3. $A \Leftrightarrow B$

Problem 1.6: Prove the following tautologies (i.e. terms which are always true):

1. $A \vee \overline{A}$

2. $(A \Rightarrow B) \Leftrightarrow \overline{A \wedge \overline{B}}$
3. $(A \Leftrightarrow B) \Leftrightarrow ((A \wedge B) \vee (\overline{A} \wedge \overline{B}))$

Problem 1.7: Express the following statements with quantifiers:

1. "For all problems there exists an solution."
2. "Not all people with a driving license own a car."
3. "There are passengers in public transport who do not have a valid ticket."

Problem 1.8: Express the statements of the previous problem with negated fragments in plain words and with quantifiers.

Problem 1.9: For the following sets A and B find out which terms are correct:

set A set B	$\{1,2,3\}$ $\{1,2,3,4\}$	$\{a,b,c,d\}$ $\{a,b,d\}$	$\mathbb{N}$ $\mathbb{N}$	$\mathbb{N}$ $\mathbb{Z}$
$A = B$				
$A \neq B$				
$A \subseteq B$				
$A \subset B$				
$A \supseteq B$				
$A \supset B$				

Problem 1.10: With $A = \{1,2,3,a,b,c\}$, $B = \{1,2,3,4,5\}$ and $C = \{a,b,c,d,e\}$ give the result of the following terms by extensional definition:

1. $A \cup B$
2. $A \cap B$
3. $A \cap C$
4. $A \setminus B$
5. $(B \cup C) \setminus A$

Problem 1.11: Proof the first and second distributive law for conjunction and disjunction.

Problem 1.12: With $A = \{1,2,3\}$ and $B = \{a,b,c\}$ write the Cartesian product $A \times B$ by extensional definition.

Problem 1.13: With $A = \{0,1\}$ define extensional the Cartesian product $A \times A \times A$.

Problem 1.14: Write the following functions as a subset of the Cartesian product by extensional definition:

1. $f : \left\{ \begin{array}{l} \mathbb{N} \to \mathbb{N} \\ x \mapsto 2x+1 \end{array} \right.$
2. $f : \left\{ \begin{array}{l} \mathbb{Z} \to \mathbb{Z} \\ x \mapsto x^2 \end{array} \right.$

Problem 1.15: With $A = \{1,2,3\}$ and $R \subseteq A \times A$ check for Cartesian product, binary relation and function:

subset $R \subseteq A \times A$	Cartesian product	binary relation	function
$\{(1,1),(2,2),(3,3)\}$			
$\{(1,1),(1,2),(1,3),$ $(2,1),(2,2),(2,3),$ $(3,1),(3,2),(3,3)\}$			
$\{(1,1),(2,1),(2,2),$ $(3,2),(3,3)\}$			
$\{(1,3),(2,2),(3,1)\}$			
$\{(1,2),(2,2),(3,2)\}$			

Problem 1.16: Check the following terms for injectivity, surjectivity and bijectivity:

term	injective	surjective	bijective
$f : \left\{ \begin{array}{l} \mathbb{N} \to \mathbb{N} \\ x \mapsto 3x+1 \end{array} \right.$			
$f : \left\{ \begin{array}{l} \mathbb{Z} \to \mathbb{N} \cup \{0\} \\ x \mapsto x^2 \end{array} \right.$			
$f : \left\{ \begin{array}{l} \mathbb{N} \to \mathbb{Z} \\ x \mapsto x \end{array} \right.$			
$f : \left\{ \begin{array}{l} \mathbb{Z} \to \mathbb{Z} \\ x \mapsto x \end{array} \right.$			
$f : \left\{ \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 \end{array} \right.$			
$f : \left\{ \begin{array}{l} \mathbb{R} \to \mathbb{R}_{\geq 0} \\ x \mapsto \exp(x) \end{array} \right.$			
$f : \left\{ \begin{array}{l} \mathbb{R}_{>0} \to \mathbb{R} \\ x \mapsto \ln(x) \end{array} \right.$			

2 Natural numbers, and integers

2.1 Natural numbers $\mathbb{N}$

Definition 2.1 (Natural numbers)**.** We call the set of integers larger than zero *natural numbers*

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

with the two operators '+' and '·' for *addition* and *multiplication*, respectively. ◁

Theorem 2.2 (Calculation rules for $\mathbb{N}$)**.** For $a, b, c \in \mathbb{N}$ we have:

1. Closure:
$$a + b \in \mathbb{N}$$
$$a \cdot b \in \mathbb{N}$$
2. Associativity:
$$(a + b) + c = a + (b + c)$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
3. Commutativity:
$$a + b = b + a$$
$$a \cdot b = b \cdot a$$
4. Distributivity over +:
$$a \cdot (b + c) = a \cdot b + a \cdot c$$
5. Neutral element for ·:
$$a \cdot 1 = a$$

◁

Remark: In *logic* the distributivity holds over both operators, *conjunction* and *disjunction*. For *sets* the distributivity holds over *union* and *intersection*. For the natural numbers the distributivity holds only over *addition* and not over *multiplication*.

Definition 2.3 (Divisor)**.** A natural number n is called *divisor* of $m \in \mathbb{N}$ if there exits a natural number p such that

$$m = n \cdot p$$

In short we write $n \mid m$. ◁

Remark: Every natural number n has at least the two *trivial divisors* 1 and n. The other divisors are called *non-trivial divisors*.

Natural numbers divisible by 2 are called *even*. Natural numbers not divisible by 2 are called *odd*.

Definition 2.4 (Multiple)**.** A natural number n is called a *multiple* of $m \in \mathbb{N}$ if there exists a natural number p such that

$$n = p \cdot m$$

◁

Remark: If n is a multiple of m with $n = p \cdot m$ then p is a divisor of n. Since 1 is a natural number any natural number n is a multiple of itself: $n = 1 \cdot n$.

Definition 2.5 (Prime number)**.** A natural number n larger than 1 is called *prime* if it has only the divisors 1 and n. ◁

Example 2.1. The set of the first ten prime numbers: $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$ ◁

Definition 2.6 (Prime factor)**.** Let n be a natural number. We call $p \in \mathbb{N}$ a *prime factor* of n if it is prime and a non-trivial divisor of n. ◁

Theorem 2.7 (Unique prime factorization)**.** Any natural number greater than one can be written as a unique product of primes. ◁

Example 2.2.

- $21 = 3 \cdot 7$
- $60 = 2 \cdot 2 \cdot 3 \cdot 5$
- $37 = 37$
- $999\,999 = 3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$

◁

Definition 2.8 (Product of sequences)**.** For $m, n \in \mathbb{N}$ and $m \leq n$ let a_k be a number for all $k \in \{m, m+1, \ldots, n-1, n\}$. We then write the product of all a_k:

$$\prod_{k=m}^{n} a_k := a_m \cdot a_{m+1} \cdot \ldots \cdot a_{n-1} \cdot a_n$$

For $m > n$ we define:

$$\prod_{k=m}^{n} a_k := 1 \qquad \triangleleft$$

Remark: The product of sequences may be used to express prime factorization. With $p_1, p_2, \ldots, p_n$ for n prime factors we can write

$$\prod_{k=1}^{n} p_k = p_1 \cdot p_2 \cdot \ldots \cdot p_{n-1} \cdot p_n$$

Definition 2.9 (Greatest common divisor)**.** With $m, n \in \mathbb{N}$ we call $p \in \mathbb{N}$ the *greatest common divisor* $p = \gcd(m, n)$ if

1. p is a common divisor of m and n, i.e.

$$p \mid m \quad \wedge \quad p \mid n$$

2. any other possible common divisor of m and n is smaller than p.

$\triangleleft$

Example 2.3.

- $\gcd(12, 8) = 4$
- $\gcd(9, 162) = 9$
- $\gcd(45, 49) = 1$
- $\gcd(8638, 7404) = 1234$

$\triangleleft$

Remark: The greatest common divisor can be used to cancel down fractions.

Example 2.4.

- $\frac{12}{8} = \frac{3 \cdot 4}{2 \cdot 4} = \frac{3}{2}$
- $\frac{9}{162} = \frac{1 \cdot 9}{18 \cdot 9} = \frac{1}{18}$
- $\frac{45}{49} = \frac{45 \cdot 1}{49 \cdot 1} = \frac{45}{49}$
- $\frac{8638}{7404} = \frac{7 \cdot 1234}{6 \cdot 1234} = \frac{7}{6}$

$\triangleleft$

Remark: The greatest common divisor p of two natural numbers m and n can be derived by the common prime factors. I.e. the prime factors of m and n are listed side by side and the common primes are then multiplied which leads to the greatest common divisor.

Example 2.5. What is the greatest common divisor of 180 and 168?

$$\begin{aligned} 180 &= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 2^2 \cdot 3^2 \cdot 5^1 \cdot 7^0 \\ 168 &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 2^3 \cdot 3^1 \cdot 5^0 \cdot 7^1 \\ \gcd &= 2^2 \cdot 3^1 \cdot 5^0 \cdot 7^0 = 12 \end{aligned}$$

$\triangleleft$

Definition 2.10 (Least common multiple)**.** Let m and n be natural numbers. We call $p \in \mathbb{N}$ the *least common multiple* $p = \operatorname{lcm}(m, n)$ if

1. p is a common multiple of m and n, i.e. m and n are both divisors of p:

$$m \mid p \quad \wedge \quad n \mid p$$

2. any other common multiple of m and n is greater than p.

$\triangleleft$

Example 2.6.

- $\operatorname{lcm}(12, 8) = 24$
- $\operatorname{lcm}(9, 162) = 162$
- $\operatorname{lcm}(45, 49) = 2\,205$
- $\operatorname{lcm}(8\,638, 7\,404) = 51\,828$

$\triangleleft$

Remark: The least common multiple is useful when adding fractions.

Example 2.7.

$$\frac{1}{12} + \frac{1}{8} = \frac{1 \cdot 2}{12 \cdot 2} + \frac{1 \cdot 3}{8 \cdot 3} = \frac{2+3}{24} = \frac{5}{24}$$

$\triangleleft$

Remark: The least common multiple p of two natural numbers m and n can be derived by the product of their prime factors where the common prime factors are only included once.

Example 2.8. What is the least common multiple of 180 and 168?

$$\begin{aligned}180 &= 2\cdot 2\cdot 3\cdot 3\cdot 5 = 2^2\cdot 3^2\cdot 5^1\cdot 7^0\\ 168 &= 2\cdot 2\cdot 2\cdot 3\cdot 7 = 2^3\cdot 3^1\cdot 5^0\cdot 7^1\\ \mathrm{lcm} &= 2^3\cdot 3^2\cdot 5^1\cdot 7^1 = 2\,520\end{aligned}$$

◁

Theorem 2.11 (Product of gcd and lcm)**.** Let m and n be natural numbers, $\gcd(m,n)$ the greatest common divisor and $\mathrm{lcm}(m,n)$ the least common multiple of m and n. We then have

$$\gcd(m,n)\cdot \mathrm{lcm}(m,n) = m\cdot n$$

◁

Example 2.9. With $m = 180$ and $n = 168$ from the previous examples we have

$$\begin{aligned}\gcd(m,n)\cdot \mathrm{lcm}(m,n) &= m\cdot n\\ 12\cdot 2520 &= 180\cdot 168\\ 30\,240 &= 30\,240\end{aligned}$$

◁

Theorem 2.12 (Division with remainder)**.** Let a and b be natural numbers, then there exist unique numbers $q \in \mathbb{N}\cup\{0\}$ and $r \in \{0,1,2,\ldots,b-1\}$ with

$$a = q\cdot b + r$$

We call a the *dividend*, b the *divisor*, q the *quotient* and r the *remainder*, i.e.
dividend = quotient · divisor + remainder ◁

Example 2.10.

- $10 = 2\cdot 4 + 2$
- $15 = 2\cdot 6 + 3$
- $149 = 2\cdot 50 + 49$
- $128 = 8\cdot 16 + 0$

◁

Theorem 2.13 (Euclidean algorithm)**.** For two natural numbers m and n we can find the greatest common divisor $\gcd(m,n)$ by a series of divisions with remainder:

We call the larger of the two numbers a_0 and the smaller a_1. Now we perform a division of a_0 by a_1. If the remainder is zero then a_1 is the gcd, if not we call the remainder a_2 and we divide a_1 by a_2.

This is repeated until there is no remainder left. The last divisor then is the greatest common divisor $\gcd(m,n)$. ◁

Example 2.11. Find the greatest common divisor of 276 and 192.

$$\begin{aligned}276 &= 1\cdot 192 + 84\\ 192 &= 2\cdot 84 + 24\\ 84 &= 3\cdot 24 + 12\\ 24 &= 2\cdot \underbrace{12}_{\text{gcd}} + 0\end{aligned}$$

Hence $\gcd(276,192) = 12$. ◁

2.2 Representation of natural numbers

Yet, when representing a particular natural number we used the decimal number system. However, this is just one of many ways to note numbers.

Example 2.12. *Roman numerals.*
The Romans used a number system with the following characters:

character	value
I	1
V	5
X	10
L	50
C	100
D	500
M	1000
…	…

(The Romans did not have a character for *zero*!) In principle the characters could be placed in any order, however, they are ordered with decreasing values. So, the number 27 may be written as

$$27 = 2\cdot X + 1\cdot V + 2\cdot I = XXVII$$

To avoid four equal characters in a row (e.g. $4 = IIII$) one of the characters is written before the one with the next higher value (e.g. $4 = IV$). Some examples:

$$\begin{aligned} 19 &= \cancel{XVIIII} = XVIV \\ 54 &= LIV \\ 99 &= \cancel{IC} = LXLVIV \\ 1749 &= MDCCXLVIV && (\text{* J. W. Goethe}) \\ 1750 &= MDCCL && (\text{† J. S. Bach}) \end{aligned}$$

If you try to perform multiplication or other mathematical operations with this number system you will understand why other number system superseded the Roman numerals. ◁

Definition 2.14 (Sum of sequences)**.**
For $m, n \in \mathbb{N}$ and $m \leq n$ let a_k be a number for all $k \in \{m, m+1, \dots, n-1, n\}$. We then write the sum of all a_k:

$$\sum_{k=m}^{n} a_k := a_m + a_{m+1} + \ldots + a_{n-1} + a_n$$

For $m > n$ we define:

$$\sum_{k=m}^{n} a_k := 0$$

◁

Definition 2.15 (Positional notation)**.** A *positional system* represents a natural number p to a base b with N digits $b_k, k = 0, \dots, N-1$, each being a natural number $0, 1, \dots, b-1$ with

$$p = \sum_{k=0}^{N-1} b_k \cdot b^k$$

The digits are written with decreasing order with no spaces between:

$$b_{N-1}b_{N-2}\cdots b_2b_1b_0$$

To explicitly state the base of a number the base may by written as a subscript behind the number (the brackets are optional):

$$p = (\ldots b_3b_2b_1b_0)_b$$

◁

Example 2.13. Below the numbers 0 to 20 in some number systems: decimal (base 10), binary (base 2), quinary (base 5), octal (base 8) and hexadecimal (base 16).

decimal	binary	quinary	octal	hexadecimal
0	0	0	0	0
1	1	1	1	1
2	10	2	2	2
3	11	3	3	3
4	100	4	4	4
5	101	10	5	5
6	110	11	6	6
7	111	12	7	7
8	1000	13	10	8
9	1001	14	11	9
10	1010	20	12	A
11	1011	21	13	B
12	1100	22	14	C
13	1101	23	15	D
14	1110	24	16	E
15	1111	30	17	F
16	10000	31	20	10
17	10001	32	21	11
18	10010	33	22	12
19	10011	34	23	13
20	10100	40	24	14

Except for the quinary all other number systems play an important role in mathematics and computer science. ◁

Example 2.14. Conversion from base b to base 10. We apply the sum in definition 2.15, e.g.

- $101_2 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 5_{10}$
- $123_8 = 1 \cdot 8^2 + 2 \cdot 8^1 + 3 \cdot 8^0 = 83_{10}$
- $6F_{16} = 6 \cdot 16^1 + 15 \cdot 16^0 = 111_{10}$

◁

Example 2.15. Conversion from base 10 to base b. We repeatedly perform a rest division by b until we get a quotient of zero. The rest-values give the number with base b in reverse order.

- $1\,234_{10} \to$ hexadecimal:

$$\left.\begin{array}{rl} 1\,234 = & 77 \cdot 16 + \ \ 2 \\ 77 = & \ \ 4 \cdot 16 + 13\ (D) \\ 4 = & \ \ 0 \cdot 16 + \ \ 4 \end{array}\right\uparrow = 4\text{D}2_{16}$$

- $25_{10} \to$ binary:

$$\left.\begin{array}{r@{\;=\;}r} 25 & 12\cdot 2+1\\ 12 & 6\cdot 2+0\\ 6 & 3\cdot 2+0\\ 3 & 1\cdot 2+1\\ 1 & 0\cdot 2+1 \end{array}\;\right\uparrow = 11001_2$$

◁

Example 2.16. Conversion between base 2 and 8. One octal digit gives three binary digits and vice versa. Hence, we only need to memorize the first eight rows of the table in example 2.13 to rapidly convert between the two bases.

- $1010101_2 = 1\ 010\ 101 = 125_8$
- $765_8 = 111\ 110\ 101 = 111110101_2$

◁

Example 2.17. Conversion between base 2 and 16. One hexadecimal digit gives four binary digits and vice versa. Hence, we only need to memorize the first sixteen rows of the table in example 2.13 to rapidly convert between the two bases.

- $111000111000_2 = 1110\ 0011\ 1000 = E38_{16}$
- $123_{16} = 0001\ 0010\ 0011 = 100100011_2$

Binary numbers are often grouped by four digits which make them more readable. ◁

2.3 Mathematical induction

Example 2.18. Carl Friedrich Gauß being a schoolboy nine years old once was given the task to add the first 100 natural numbers. The teacher hardly finished the sentence when little Carl wrote 5 050 on his tablet. To add the first n numbers he used the following relation:

$$\sum_{i=1}^{n} i = \frac{n\cdot(n+1)}{2}$$

Is the equation correct? Does it hold for any number of summands? How do we prove that this equation holds for any natural number n?

Carl Friedrich Gauß (1777-1855)

◁

Definition 2.16 (Definition by induction)**.** Let $n_0 \in \mathbb{N}$ and a_n be an object for all $n \geq n_0$. A way to define all a_n is

1. Extensional definition of the first object a_0 or first few objects.
2. Definition of any subsequent object by its predecessors.

We call this *definition by induction.* ◁

Example 2.19. Factorial numbers
1. $a_0 := 1$
2. $a_n := n\cdot a_{n-1}$ ◁

Example 2.20. Fibonacci number
1. $a_0 := 0$, $a_1 := 1$
2. $a_{n+1} := a_n + a_{n-1}$ ◁

Example 2.21. *Heron's method* for $\sqrt{2}$
1. $a_1 := 1$
2. $a_{n+1} := \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$

a_n converges for increasing n towards $\sqrt{2}$. ◁

Theorem 2.17 (Proof by mathematical induction)**.** Let n_0 be a natural number and $A(n)$ be a statement for all $n \geq n_0$. To prove $A(n)$ for all $n \geq n_0$ it is enough to show that

1. $A(n_0)$ is true
2. for any $n \geq n_0$ if $A(n)$ is true then $A(n+1)$ is true, i.e. $A(n) \Rightarrow A(n+1)$

We call the first *induction start* and the second *induction step.* ◁

Example 2.22. We want to prove the Gaussian sum:

$$\sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$$

1. Induction start (A_1):

$$\begin{aligned} \sum_{i=1}^{1} i &= \frac{1 \cdot (1+1)}{2} \\ 1 &= 1 \end{aligned}$$

2. Induction step ($A_n \Rightarrow A_{n+1}$): If

$$\underbrace{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}_{A_n}$$

then

$$\underbrace{\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}}_{A_{n+1}}.$$

Now the proof:

$$\begin{aligned} \sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

◁

Example 2.23. We want to prove:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

1. Induction start (A_1):

$$\begin{aligned} \sum_{i=1}^{1} i^2 &= \frac{1(1+1)(2 \cdot 1+1)}{6} \\ 1^2 &= \frac{1 \cdot 2 \cdot 3}{6} \\ 1 &= 1 \end{aligned}$$

b) Induction step ($A_n \Rightarrow A_{n+1}$): If

$$A_n : \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

then

$$A_{n+1} : \qquad \sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

Proof:

$$\begin{aligned} \sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^{n} i^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\ &= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\ &= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

◁

Theorem 2.18 (Bernoulli's inequality)**.** For $b \in \mathbb{R}$, $b \geq -1$ and $n \in \mathbb{N}$ we have:

$$(1+b)^n \geq 1 + n \cdot b$$

◁

Proof. 1. Induction start (A_1):

$$\begin{aligned} (1+b)^1 &\geq 1 + 1 \cdot b \\ 1 + b &\geq 1 + b \end{aligned}$$

2. Induction step ($A_n \Rightarrow A_{n+1}$): If

$$(1+b)^n \geq 1 + n \cdot b$$

then

$$(1+b)^{n+1} \geq 1 + (n+1)b$$

and we show

$$\begin{aligned} (1+b)^{n+1} &= (1+b)^n(1+b) \geq (1+nb)(1+b) \\ &= 1 + \underbrace{b + nb}_{(n+1)b} + \underbrace{nb^2}_{\geq 0} \geq 1 + (n+1)b \end{aligned}$$

□

Theorem 2.19 (Geometric sum)**.** For $q \in \mathbb{R}$, $q \neq 1$ and $n \in \mathbb{N}$ we have

$$\sum_{k=0}^{n-1} q^k = \frac{1 - q^n}{1 - q}$$

◁

Proof. 1. Induction start (A_1):

$$\begin{aligned}\sum_{k=0}^{1-1} q^k &= \frac{1-q^1}{1-q}\\ q^0 &= \frac{1-q}{1-q}\\ 1 &= 1\end{aligned}$$

2. Induction step ($A_n \Rightarrow A_{n+1}$): If

$$\sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q}$$

then

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}.$$

and we show

$$\begin{aligned}\sum_{k=0}^{n} q^k = \sum_{k=0}^{n-1} q^k + q^n &= \frac{1-q^n}{1-q} + q^n\\ &= \frac{1-q^n+q^n-q\cdot q^n}{1-q} = \frac{1-q^{n+1}}{1-q}\end{aligned}$$

□

2.4 Integers $\mathbb{Z}$

Sums and products of natural numbers are again natural numbers. In dealing with engineering problems, one very frequently has to solve equations. Unfortunately the equation

$$n + x = m$$

does not have a solution $x \in \mathbb{N}$ for arbitrary numbers $m, n \in \mathbb{N}$.

If we want to solve equations of the above type without limitations, we are led to the system of integers $\mathbb{Z}$: For each $n \in \mathbb{N}$ a new *negative* number $-n$ is provided in our new number system and in addition to that a further number called 0 (zero) is introduced. This new number system is called the set of *integers* denoted by $\mathbb{Z}$. Obviously $\mathbb{N} \subset \mathbb{Z}$.

If we denote the set of *negative* numbers by $\mathbb{N}'$ then we have the following situation:

- If $a \in \mathbb{Z} \setminus 0$ then either $a \in \mathbb{N}$ or $a \in \mathbb{N}'$
- If $a \in \mathbb{N}$ then there is $a' \in \mathbb{N}'$ such that $a + a' = 0$

Theorem 2.20 (Calculation rules for $\mathbb{Z}$)**.** For $a, b, c \in \mathbb{Z}$ we have:

1. Closure:

$$\begin{aligned}a + b &\in \mathbb{Z}\\ a \cdot b &\in \mathbb{Z}\end{aligned}$$

2. Associativity:

$$\begin{aligned}(a + b) + c &= a + (b + c)\\ (a \cdot b) \cdot c &= a \cdot (b \cdot c)\end{aligned}$$

3. Commutativity:

$$\begin{aligned}a + b &= b + a\\ a \cdot b &= b \cdot a\end{aligned}$$

4. Distributivity over +:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

5. Neutral element for + and ·:

$$\begin{aligned}a + 0 &= a\\ a \cdot 1 &= a\end{aligned}$$

6. Inverse element for +: There exists an inverse element $a' \in \mathbb{Z}$ such that

$$a + a' = 0$$

7. If $a \cdot b = 0$ then either $a = 0$ or $b = 0$:

$$a \cdot b = 0 \quad \leftrightarrow \quad ((a = 0) \vee (b = 0))$$

◁

Remark: Unfortunately, equations of the form

$$m \cdot x = n$$

have no solution in $\mathbb{Z}$ for arbitrary $m, n \in \mathbb{Z}$ which leads us to the rational numbers $\mathbb{Q}$ in the next section.

Example 2.24. There exists no $x \in \mathbb{Z}$ that satisfies the following equation:

$$2 \cdot x = 3$$

◁

2.5 Problems

Problem 2.1: In the context of natural numbers explain the following terms:

1. divisor
2. *trivial* and *non-trivial* divisor
3. multiple
4. prime number
5. prime factor

Problem 2.2: Perform a prime factorization on the following numbers:

1. 123
2. 321
3. 30 030
4. 65 536
5. 9 000 000

Problem 2.3: Simplify the following terms:

1. $\prod_{i=1}^{5} 2$
2. $\prod_{i=1}^{4} i$
3. $\prod_{i=1}^{3} i^2$
4. $\frac{1}{2^n} \prod_{i=1}^{n} 2x$
5. $\prod_{i=1}^{n} i - n!$
6. $\frac{\prod_{i=1}^{7} f^i(x)}{\prod_{i=2}^{7} f^i(x)}$

with $f(x) \neq 0$ for all x

Problem 2.4: Evaluate the *greatest common divisor* and *least common multiple* of the following pairs:

a	b	$\gcd(a,b)$	$\operatorname{lcm}(a,b)$
6	8		
30	20		
14	15		
39	117		
123	123		

Problem 2.5: If the greatest common divisor and the least common multiple of two natural numbers a and b are known, is it possible to evaluate a and b?

Problem 2.6: Complete out the following table:

dividend	divisor	quotient	remainder
100	16		
62	7		
156	26		
123	124		
124	123		
1 000	3		

Problem 2.7: Perform the Euclidean algorithm to the following pairs of natural numbers to find the greatest common divisor:

1. 80 and 115
2. 768 and 540
3. 5 471 and 1 193

Problem 2.8: Change the base of the following representations:

1. $100_{10} \rightarrow$ binary
2. $100_{10} \rightarrow$ octal
3. $100_{10} \rightarrow$ decimal
4. $100_{10} \rightarrow$ hexadecimal
5. $100_{2} \rightarrow$ decimal
6. $100_{8} \rightarrow$ decimal
7. $100_{10} \rightarrow$ decimal
8. $100_{16} \rightarrow$ decimal

Problem 2.9: Fill out the following table:

binary	octal	hexadecimal
11 0111		
1010 0101 1100		
	17	
	6543	
		123
		ABC

Problem 2.10: Prove by mathematical induction:

1. $\sum_{i=1}^{n} i^3 = \frac{n^2 \cdot (n+1)^2}{4}$
2. $n^3 + 2n$ has the divisor 3
3. $n! > 2^n$ for $n > 3$

3 Rational and real numbers

3.1 Rational numbers $\mathbb{Q}$

Definition 3.1 (Rational numbers)**.** We define the set of *rational numbers* $\mathbb{Q}$ as the quotient of any $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, i.e.

$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p \in \mathbb{Z}, q \in \mathbb{N} \right\}$$

◁

Example 3.1.

$$\begin{array}{rrr} \frac{1}{2} \in \mathbb{Q} & -\frac{3}{7} \in \mathbb{Q} & \frac{48}{64} = \frac{3}{4} \in \mathbb{Q} \\ \frac{5}{2} = 1\frac{1}{2} \in \mathbb{Q} & 1 \in \mathbb{Q} & 0 \in \mathbb{Q} \\ \sqrt{2} \notin \mathbb{Q} & \pi \notin \mathbb{Q} & e \notin \mathbb{Q} \end{array}$$

◁

Theorem 3.2 (Properties of rational numbers)**.** For any $a, b, c \in \mathbb{Q}$ and $+$ and $\cdot$ as commonly used we have:

1. Closure:

$$a + b \in \mathbb{Q}$$
$$a \cdot b \in \mathbb{Q}$$

2. Associativity:

$$(a + b) + c = a + (b + c)$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

3. Commutativity:

$$a + b = b + a$$
$$a \cdot b = b \cdot a$$

4. Neutral element for $+$ and $\cdot$:

$$a + 0 = a$$
$$a \cdot 1 = a$$

5. Inverse element: There exists the inverse elements $a', b' \in \mathbb{Q}$ such that

$$a + a' = 0$$
$$b \cdot b' = 1 \qquad \text{for } b \neq 0$$

   where 0 and 1 denote the neutral elements for $+$ and $\cdot$, respectively.

6. Distributivity over $+$:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

7. If $a \cdot b = 0$ then either $a = 0$ or $b = 0$ and vice versa, i.e.

$$a \cdot b = 0 \quad \Leftrightarrow \quad a = 0 \lor b = 0$$

◁

Remark: We denote the inverse element for addition with a minus sign.

$$a' = -a$$

In short we write:

$$a + (-b) \quad \Leftrightarrow \quad a - b$$

We denote the inverse element for multiplication with an exponent of -1 or with b in the denominator of a fraction.

$$b' = b^{-1} = \frac{1}{b}$$

Definition 3.3 (Field)**.** We call a set with the properties shown in theorem 3.2 a *Field.* ◁

Remark: For a set A with the two operators $+$ and $\cdot$ we write:

$$(A, +, \cdot)$$

The symbols $+$ and $\cdot$ may also stand for other operators as shown in the next example.

Example 3.2. Boolean field.

We define the set $B = \{0, 1\}$ with the two operators

$+$	0	1
0	0	1
1	1	0

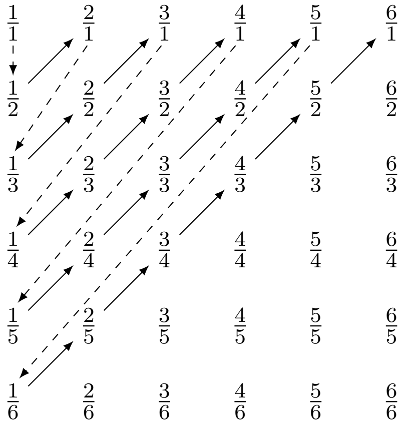
$\cdot$	0	1
0	0	0
1	0	1

(In terms of *logic* '+' is the *exclusive disjunction* and '·' equals the *conjunction.*) The neutral elements for '+' and '·' are 0 and 1, respectively.

It can be shown that the set B forms a field by proving all properties of theorem 3.2. This may be done by truth tables as we did in an earlier section. ◁

Theorem 3.4 (ℚ is countable)**.** The rational numbers $\mathbb{Q}$ are countable. ◁

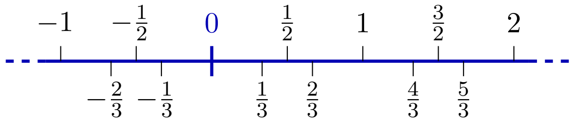
Proof. We arrange the elements of $\mathbb{Q}$ in a table:



and we write:

$$\mathbb{Q} = \{0, \pm\tfrac{1}{1}, \pm\tfrac{1}{2}, \pm\tfrac{2}{1}, \pm\tfrac{1}{3}, \pm\tfrac{2}{2}, \pm\tfrac{3}{1}, \ldots\}$$ □

Remark: We can write rational numbers on a real line:



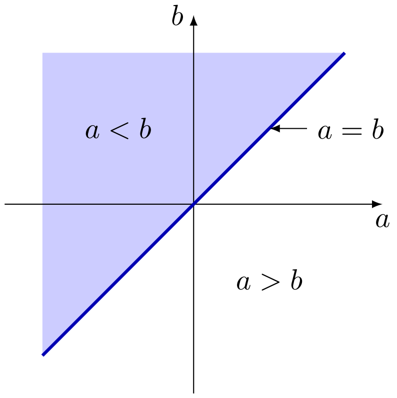
By this visualization each number has a unique position on the line.

Definition 3.5 (Less or equal)**.** For $a, b \in \mathbb{Q}$ we say a is *less or equal* b if a is not on the right of b on the real line. In short we write

$$a \leq b$$

and also call it an *order relation*.

The relations $\geq$, $<$ and $>$ are defined in a similar manner. ◁



Definition 3.6 (Totally ordered sets)**.** Let A be a set with the order relation $\leq$ and the following properties:

1. comparability: For all $a, b \in A$
$$a \leq b \quad \text{or} \quad b \leq a$$
2. reflexivity: For all $a \in A$
$$a \leq a$$
3. antisymmetry: For all $a, b \in A$
$$a \leq b \text{ and } b \leq a \quad \Leftrightarrow \quad a = b$$
4. transitivity: For all $a, b, c \in A$
$$a \leq b \quad \text{and} \quad b \leq c \quad \text{implies} \quad a \leq c$$

We call such a set a *totally ordered set.* ◁

Example 3.3. The set of rational numbers $\mathbb{Q}$ is an ordered set. All properties of definition 3.6 hold. ◁

Example 3.4. The set of all surnames of an address book with an order relation $\leq$ to sort the names is an ordered set. ◁

Example 3.5. Let $A = \mathbb{R} \times \mathbb{R}$ be the set of all coordinates on a plane. We define the order relation $\leq$ on the distance to the origin, i.e. $r = \sqrt{x^2 + y^2}$, $(x, y) \in A$.

The order relation is not antisymmetric: E.g. for $r_1 = (1, 1)$ and $r_2 = (1, -1)$ we have $r_1 \leq r_2$ and $r_2 \leq r_1$ but $(1, 1) \neq (1, -1)$.

Hence, A is *not* an totally ordered set. ◁

Theorem 3.7 (Properties of totally ordered fields)**.** A totally ordered set A with arbitrary elements $a, b, c \in A$ has the following properties:

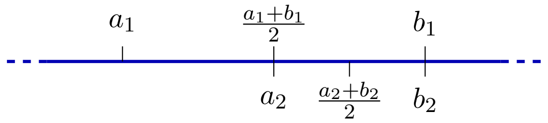
- $a < b \quad \text{or} \quad a = b \quad \text{or} \quad a > b$ (trichotomy)
- $a > 0 \wedge b > 0 \quad \Rightarrow \quad a \cdot b > 0$
- $a < b \quad \Leftrightarrow \quad a + c < b + c$
- $a < b \wedge c > 0 \quad \Rightarrow \quad c \cdot a < c \cdot b$
- $a < b \wedge c < 0 \quad \Rightarrow \quad c \cdot a > c \cdot b$
- $a < b \quad \Leftrightarrow \quad (-a) > (-b)$
- $a \neq 0 \quad \Leftrightarrow \quad a^2 > 0$

- $a > 0 \quad \Leftrightarrow \quad a^{-1} > 0$

◁

Theorem 3.8 ($\mathbb{Q}$ is dense)**.** For every $a, b \in \mathbb{Q}$ with $a < b$ there exists a $c \in \mathbb{Q}$ with $a < c < b$. I.e. for any two unequal rational numbers there exists an infinite number of rational numbers between them. ◁

Proof. We start with two rational numbers $a_1, b_1 \in \mathbb{Q}$ where $a_1 < b_1$. The term $\frac{a_1+b_1}{2}$ is greater than a_1, less than b_1 and again is a rational number. We now either set $a_2 = a_1$ and $b_2 = \frac{a_1+b_1}{2}$ or $a_2 = \frac{a_1+b_1}{2}$ and $b_2 = b_1$. Again the term $\frac{a_2+b_2}{2}$ is greater than a_2, less than b_2 and again is a rational number. This may be repeated infinite times for any position on the real line. □



Remark: Although the rational numbers $\mathbb{Q}$ are dense there exist numbers like $\sqrt{2}$ or π that can not be expressed by rational numbers.

It is possible to approximate such numbers with arbitrary small error larger than zero by increasing numerator and denominator, however, we will never be able to express these numbers exactly.

It is possible to express these *irrational* numbers by an infinite sum of rational numbers. I.e. for π we may write

$$\pi = 4 \cdot \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right)$$

Hence, we are looking for an extended set of numbers that include irrational numbers.

3.2 Real numbers $\mathbb{R}$

Definition 3.9 (Irrational numbers)**.** Let S be the set of infinite sums of rational numbers. We call the set of all elements $s \in S$ that are not rational the set of *irrational numbers.* ◁

Example 3.6. Some irrational numbers:

- $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} - \ldots = e$
- $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots = \frac{1}{e}$
- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \ln 2$
- $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots = \frac{\pi}{4}$

◁

Definition 3.10 (Real numbers)**.** We call the set of all possible rational number sums, finite or infinite, the set of *real numbers* $\mathbb{R}$. ◁

Theorem 3.11 (Properties of $\mathbb{R}$)**.** The set of real numbers $\mathbb{R}$ has the following properties:

- $\mathbb{R}$ is a field
- $\mathbb{R}$ is not countable
- $\mathbb{R}$ is totally ordered
- $\mathbb{R}$ is dense

◁

Remark: These four properties are stated without proof. However, in problem 3.4 and 3.4 you are asked to prove two of them.

3.3 Intervals and absolute values

Definition 3.12 (Finite intervals)**.** For $a, b \in \mathbb{R}$ and $a < b$ we define *finite intervals*:

- *closed interval*:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

- *open interval*:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

- *left open interval*:

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

- *right open interval*:

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

◁

Definition 3.13 (Infinite intervals)**.** For $a \in \mathbb{R}$ we define *infinite intervals*:

$$\begin{aligned}
[a, \infty) &= \{x \in \mathbb{R} \mid a \leq x < \infty\} \\
(a, \infty) &= \{x \in \mathbb{R} \mid a < x < \infty\} \\
(-\infty, a] &= \{x \in \mathbb{R} \mid -\infty < x \leq a\} \\
(-\infty, a) &= \{x \in \mathbb{R} \mid -\infty < x < a\}
\end{aligned}$$

◁

Example 3.7.

- $\mathbb{R} = (-\infty, \infty) = \{x \in \mathbb{R} \mid -\infty < x < \infty\}$
- $\mathbb{R}_{>0} = (0, \infty) = \{x \in \mathbb{R} \mid 0 < x < \infty\}$
- $\mathbb{R}_{\geq 0} = [0, \infty) = \{x \in \mathbb{R} \mid 0 \leq x < \infty\}$

◁

Definition 3.14 (Supremum and infimum)**.** Let A be an arbitrary subset of $\mathbb{R}$.

- We call the least number of $\mathbb{R} \cup \{\infty\}$ greater or equal to all elements of A the *supremum* of A or the *least upper bound* of A denoted by
$$\sup(A)$$
- We call the greatest number of $\mathbb{R} \cup \{-\infty\}$ less or equal to all elements of A the *infimum* of A or the *greatest lower bound* of A denoted by
$$\inf(A)$$

◁

Example 3.8.

- $\sup(\{1, 2, 3\}) = 3$
- $\inf(\{1, 2, 3\}) = 1$
- $\sup(\mathbb{R}) = \infty$
- $\inf(\mathbb{R}_{>0}) = 0$
- $\inf(\mathbb{R}_{\geq 0}) = 0$

◁

Definition 3.15 (Maximum and minimum)**.** Let A be an arbitrary subset of $\mathbb{R}$

- If the supremum of A is an element of A we call it the *maximum* of A denoted by
$$\max(A)$$
- If the infimum of A is an element of A we call it the *minimum* of A denoted by
$$\min(A)$$

◁

Example 3.9.

- Let $A = [-1, 1]$ then
$$\max(A) = 1 \quad \text{and} \quad \min(A) = -1$$
- Let $B = \{y \in \mathbb{R} \mid y = x^2, x \in [-1, 1]\}$ then
$$\max(B) = 1 \quad \text{and} \quad \min(B) = 0$$
- Let $C = (0, 10)$ then $\max(C)$ and $\min(C)$ are undefined!

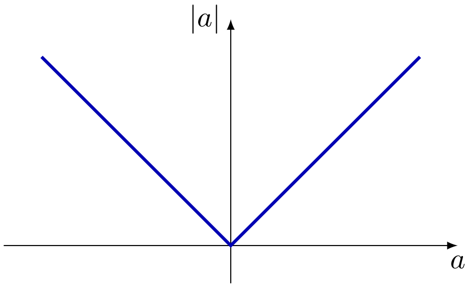
◁

Remark: The minimum and maximum exist for closed intervals and for finite sets (i.e. sets with a finite number of elements). If a maximum or minimum exists it equals the supremum or infimum, respectively.

Definition 3.16 (Absolute value)**.** For $a \in \mathbb{R}$ we define the *absolute value* of a by

$$|a| := \begin{cases} a & \text{for} \quad a \geq 0 \\ -a & \text{for} \quad a < 0 \end{cases}$$

◁



Theorem 3.17 (Properties of absolute values)**.** For any $a, b \in \mathbb{R}$ we have:

- $|a| \geq 0$
- $|a| = 0 \quad \Leftrightarrow \quad a = 0$
- $|a \cdot b| = |a| \cdot |b|$
- $|a + b| \leq |a| + |b|$ (triangle inequality)

◁

3.4 Problems

Problem 3.1: Let $A = \{a, b\}$ be a set with the two operators $+$ and $\cdot$ as defined in the following tables:

+	a	b
a	a	b
b	b	a

·	a	b
a	a	a
b	a	b

Is $(A, +, \cdot)$ a field? If not, why?

Problem 3.2: Is the set of integers $\mathbb{Z}$ with the operators $+$ and $\cdot$ for addition and multiplication as commonly used a field? If not, why?

Problem 3.3: Let $A = \{a, b, c\}$ be a set with the two operators $+$ and $\cdot$ as defined in the following tables:

+	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

·	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b

Is $(A, +, \cdot)$ a field? If not, why?

Problem 3.4: Are the following sets with the order relation $\leq$ as commonly used totally ordered? If not, why?

1. natural numbers $\mathbb{N}$
2. integers $\mathbb{Z}$

Problem 3.5: Is the set of real numbers $\mathbb{R}$ with the operators $+$ and $\cdot$ for addition and multiplication as commonly used a field? If not, why?

Problem 3.6: Is the set of real numbers $\mathbb{R}$ an totally ordered field? If not, why?

Problem 3.7: Give the intervals of the following sets:

1. $A_1 = \{y \in \mathbb{R} \mid y = x^2, x \in \mathbb{R}_{>0}\}$
2. $A_2 = \{y \in \mathbb{R} \mid y = x^2, x \in \mathbb{R}_{\leq 0}\}$
3. $A_3 = \{y \in \mathbb{R} \mid y = x^{-1}, x \in \mathbb{R}_{\geq 1}\}$
4. $A_4 = \{y \in \mathbb{R} \mid y = \exp(x), x \in \mathbb{R}\}$
5. $A_5 = \{y \in \mathbb{R} \mid y = \sin(x), x \in \mathbb{R}\}$

Problem 3.8: Find the supremum and infimum of the following sets.

set	sup	inf
$A_1 = \{1, 2, 3, 4\}$		
$A_2 = \{-2, 0, 2, 4\}$		
$A_3 = \{-100, 200\}$		
$A_4 = \{3, 6, 9, \ldots\}$		
$A_5 = \{\ldots, -1, 0, 1, 2, \ldots\}$		

Problem 3.9: For the sets in problem 3.4 find the supremum, infimum, maximum and minimum.

set	sup	inf	max	min
A_1				
A_2				
A_3				
A_4				
A_5				

Problem 3.10: Give the extensional definition of the following sets:

1. $A_1 = \{n \in \mathbb{Z} \mid n = |k|, k \in \mathbb{Z}\}$
2. $A_2 = \{y \in \mathbb{R} \mid y = \frac{x}{|x|}, x \in \mathbb{R}_{\neq 0}\}$
3. $A_3 = \{n \in \mathbb{Z} \mid n = |10k - 1|, k \in \mathbb{Z}\}$

4 Complex numbers

4.1 Why complex numbers?

Let us recall the reasons to develop extended number sets.

- We start with the natural numbers $\mathbb{N}$.
- $n + x = m$ has no solution in $\mathbb{N}$ for $n \geq m$ which leads us to the integers $\mathbb{Z}$.
- $n \cdot x = m$ has no solution in $\mathbb{Z}$ for m is not a multiple of n which leads us to the rational numbers $\mathbb{Q}$.
- $x^2 - c = 0$ has no solution if c is not a square of a fraction (e.g. 2) which leads us to the real numbers $\mathbb{R}$.

Already the argument which leads to the real numbers point at the next limit to overcome: If c in equation $x^2 - c = 0$ is negative, we get to the point to take the square root of a negative number which has no solution in $\mathbb{R}$.

Hence, we are looking for a number system that gives us a solution for equations like

$$x^2 + 1 = 0$$

4.2 Definition of complex numbers

Definition 4.1 (Imaginary unit)**.** We define the constant $\mathrm{j} \notin \mathbb{R}$ by

$$\mathrm{j}^2 = -1$$

and call it *imaginary unit*. A multiple of j is called *imaginary number*. ◁

Remark: In mathematics the letter i is used for the imaginary unit. However, in electrical engineering the *i* is *the* symbol for the electrical current and, hence, we use the j as a symbol for the imaginary unit.

Example 4.1. We want to solve the equation $x^2 = -4$ for x:

$$\begin{aligned} x^2 &= -4 \\ x^2 &= 4 \cdot (-1) \\ x^2 &= 4 \cdot \mathrm{j}^2 \\ x &= (\pm 2) \cdot (\pm \mathrm{j}) \\ x &= \pm 2\,\mathrm{j} \end{aligned}$$

Towards the end of this chapter we will see that care must be taken when dealing with roots on complex numbers. However, the result $\pm 2\mathrm{j}$ in this example is correct. ◁

Theorem 4.2 (Powers of j)**.** Although j is not real, we may use it as a unit and perform mathematical operations as usual. Powers of j may be combined:

$$\begin{aligned} \mathrm{j}^1 &= \mathrm{j} \\ \mathrm{j}^2 &= -1 \qquad \text{(by definition)} \\ \mathrm{j}^3 &= (\mathrm{j}^2) \cdot \mathrm{j} = (-1) \cdot \mathrm{j} = -\mathrm{j} \\ \mathrm{j}^4 &= \mathrm{j}^2 \cdot \mathrm{j}^2 = (-1) \cdot (-1) = 1 \\ \mathrm{j}^5 &= \mathrm{j}^4 \cdot \mathrm{j} = \mathrm{j} \\ \mathrm{j}^0 &= 1 \\ \mathrm{j}^{-1} &= \tfrac{1}{\mathrm{j}} = \tfrac{\mathrm{j}^4}{\mathrm{j}} = \mathrm{j}^3 = -\mathrm{j} \end{aligned}$$

n	…	-2	-1	0	1	2	3	4	5	…
j^n	…	-1	$-\mathrm{j}$	1	j	-1	$-\mathrm{j}$	1	j	…

◁

Definition 4.3 (Complex number)**.** With $a, b \in \mathbb{R}$ we define complex numbers as $a + \mathrm{j} \cdot b$ or $a + \mathrm{j}\,b$, i.e.

$$\mathbb{C} = \{a + \mathrm{j}\,b \mid a, b \in \mathbb{R}, \mathrm{j}^2 = -1\}$$

We say two complex numbers are equal if their components a and b are equal, i.e. for $c_1 = a_1 + \mathrm{j}\,b_1$ and $c_2 = a_2 + \mathrm{j}\,b_2$ we have:

$$c_1 = c_2 \quad \Leftrightarrow \quad a_1 = a_2 \wedge b_1 = b_2$$

◁

Definition 4.4 (Complex number, alternative definition)**.** The Cartesian product $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ together with the two operators

$$+ : \begin{cases} \mathbb{C} \times \mathbb{C} \to \mathbb{C} \\ (a_1, b_1) + (a_2, b_2) \mapsto \\ \qquad (a_1 + a_2, b_1 + b_2) \end{cases}$$

$$\cdot : \begin{cases} \mathbb{C} \times \mathbb{C} \to \mathbb{C} \\ (a_1, b_1) \cdot (a_2, b_2) \mapsto \\ \qquad (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1) \end{cases}$$

is called the set of *complex numbers*. ◁

Remark: Both definitions are complete. One may be concluded from the other. In the following we use the imaginary constant j to note complex numbers and not the ordered pairs of a Cartesian product.

Definition 4.5 (Real and imaginary part)**.** For any complex number $c = a + \mathrm{j}\, b \in \mathbb{C}$ we define

$$\mathrm{Re}(c) = \mathrm{Re}(a + \mathrm{j}\, b) = a$$

as the *real part* of c and

$$\mathrm{Im}(c) = \mathrm{Im}(a + \mathrm{j}\, b) = b$$

as the *imaginary part* of c. ◁

Example 4.2.

- $\mathrm{Re}(3 - \mathrm{j}\, 4) = 3$
- $\mathrm{Im}(3 - \mathrm{j}\, 4) = -4$
- $\mathrm{Re}(\mathrm{j}) = 0$
- $\mathrm{Im}(\mathrm{j}) = 1$
- $\mathrm{Re}(\mathrm{j}^2) = -1$
- $\mathrm{Im}(5) = 0$

◁

Remark: Note that the imaginary part of a complex number is real, i.e. for any $c \in \mathbb{C}$ we have:

$$\mathrm{Im}(c) \in \mathbb{R}$$

As part of a complex number we multiply the imaginary part with j which gives us an imaginary number.

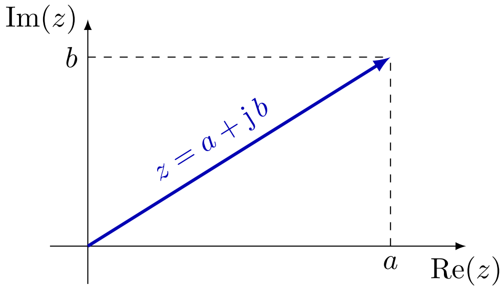
4.3 Properties of complex numbers

Theorem 4.6 (Properties of complex numbers)**.**

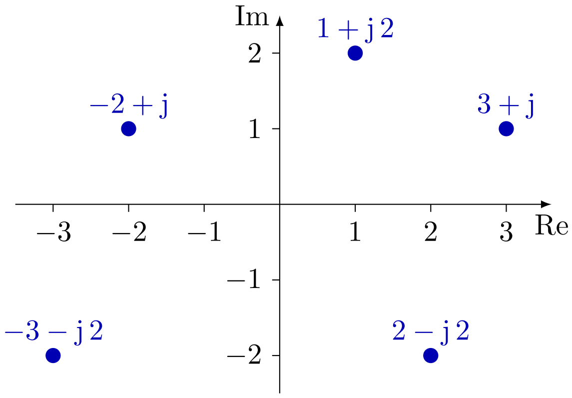
- $\mathbb{C}$ is a field
- $\mathbb{C}$ is not totally ordered
- $\mathbb{C}$ is not countable
- $\mathbb{C}$ is dense

◁

Remark: A complex number $z = a + \mathrm{j}\, b$ can be represented as a point (a, b) or an arrow to this point on a *complex plane*. The real and imaginary parts are plotted on the abscissa and ordinate, respectively.



Example 4.3.



◁

Definition 4.7 (Absolute)**.** The *absolute* of a complex number $z = a + \mathrm{j}\, b \in \mathbb{C}$ is defined by

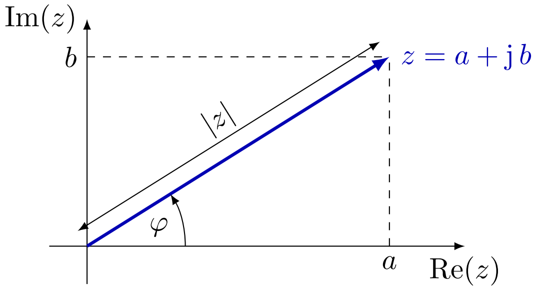
$$|z| = \sqrt{a^2 + b^2}$$

◁

Theorem 4.8 (Polar form)**.** A complex number $z = a + \mathrm{j}\, b \in \mathbb{C}$ can also be expressed by polar coordinates $|z|$ and φ where is the angle from the positive real axis counter-clockwise to the arrow on the complex plane. We have:

$$\begin{aligned} a &= |z| \cdot \cos\varphi \\ b &= |z| \cdot \sin\varphi \\ |z| &= \sqrt{a^2 + b^2} \\ \varphi &= \arg z \end{aligned}$$

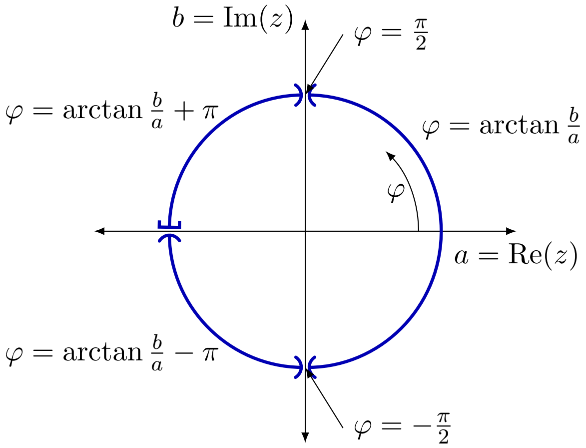
where $\arg z$ is defined as follows. ◁



Definition 4.9 (Argument)**.** With $z = a + \mathrm{j}b$, $a, b \in \mathbb{R}$ the function $\arg(z)$ maps a complex number to the interval $(-\pi, \pi]$ defined with:

$$\arg(z) = \begin{cases} \arctan \frac{b}{a} - \pi & \text{for } a < 0 \text{ and } b < 0 \\ -\frac{\pi}{2} & \text{for } a = 0 \text{ and } b < 0 \\ \arctan \frac{b}{a} & \text{for } a > 0 \\ \frac{\pi}{2} & \text{for } a = 0 \text{ and } b > 0 \\ \arctan \frac{b}{a} + \pi & \text{for } a < 0 \text{ and } b \geq 0 \\ \text{undefined} & \text{for } a = 0 \text{ and } b = 0 \end{cases}$$

◁



Remark: Now a complex number can be written with their polar coordinates.

$$z = a + \mathrm{j}\, b = |z| \cdot (\cos \varphi + \mathrm{j}\, \sin \varphi)$$

Later we will learn Euler's formula:

$$\exp(\mathrm{j}\, x) = e^{\mathrm{j}\, x} = \cos x + \mathrm{j}\, \sin x$$

which further simplifies our complex number to

$$z = a + \mathrm{j}\, b = |z| e^{\mathrm{j}\, \varphi}$$

Remark: By definition 4.9 the polar angle is an element of $(-\pi, \pi]$. Other definitions limit the angle to $[0, 2\pi)$. Due to the periodicity of the exponential function for imaginary values we have:

$$z = |z| \cdot e^{\mathrm{j} \arg(z)} = |z| \cdot e^{\mathrm{j}(2\pi + \arg(z))}$$

or more general:

$$z = |z| \cdot e^{\mathrm{j}(2\pi n + \arg(z))} \quad \text{for any } n \in \mathbb{Z}$$

Example 4.4.

- $z_1 = 4 + \mathrm{j}\, 4 = 4\sqrt{2} \cdot e^{\mathrm{j}\pi/4}$
- $z_2 = \sqrt{3} - \mathrm{j} = 2 \cdot e^{-\mathrm{j}\pi/6}$
- $z_3 = -1 - \mathrm{j} = \sqrt{2} \cdot e^{-\mathrm{j}3\pi/4} = -\sqrt{2} \cdot e^{\mathrm{j}\pi/4}$

◁

4.4 Basic operations

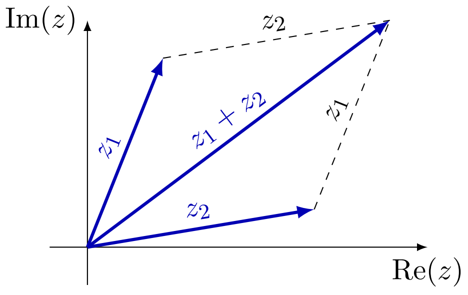
Theorem 4.10 (Basic arithmetic operations)**.** With $z_1 = a_1 + \mathrm{j}\, b_1 \in \mathbb{C}$ and $z_2 = a_2 + \mathrm{j}\, b_2 \in \mathbb{C}$ we have for the basic arithmetic operations:

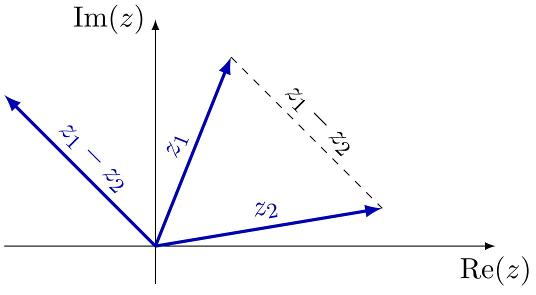
$$\begin{aligned} z_1 + z_2 &= a_1 + a_2 + \mathrm{j}(b_1 + b_2) \\ z_1 - z_2 &= a_1 - a_2 + \mathrm{j}(b_1 - b_2) \\ z_1 \cdot z_2 &= a_1 a_2 - b_1 b_2 + \mathrm{j}(a_1 b_2 + a_2 b_1) \end{aligned}$$

and for $z_2 \neq 0$:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1 + \mathrm{j}\, b_1}{a_2 + \mathrm{j}\, b_2} \cdot \frac{a_2 - \mathrm{j}\, b_2}{a_2 - \mathrm{j}\, b_2} \\ &= \frac{a_1 a_2 + b_1 b_2}{{a_2}^2 + {b_2}^2} + \mathrm{j} \frac{a_2 b_1 - a_1 b_2}{{a_2}^2 + {b_2}^2} \end{aligned}$$

◁





Example 4.5. For $z_1 = 1 + \mathrm{j}$ and $z_2 = 2 - 2\mathrm{j}$ we have

- $z_1 + z_2 = 3 - \mathrm{j}$
- $z_1 - z_2 = -1 + 3\mathrm{j}$
- $z_1 \cdot z_2 = 4$
- $\dfrac{z_1}{z_2} = \dfrac{1}{2}\mathrm{j}$

◁

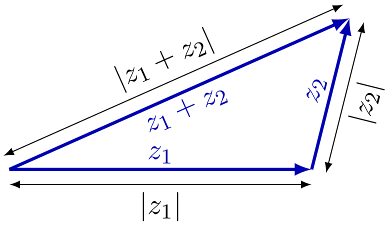
Remark: The multiplication and division of complex numbers are difficult to illustrate on the complex plane. They are better understood in polar coordinates where the absolutes are multiplied/divided and the angles are added/subtracted, respectively.

$$
\begin{aligned}
z_1 \cdot z_2 &= |z_1|\,|z_2| e^{\mathrm{j}\,(\varphi_1 + \varphi_2)} \\
\frac{z_1}{z_2} &= \frac{|z_1|}{|z_2|} e^{\mathrm{j}\,(\varphi_1 - \varphi_2)}
\end{aligned}
$$

Theorem 4.11 (Properties of absolute)**.** With $z, z_1, z_2 \in \mathbb{C}$ we have

- $|z| \geq 0$
- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$ (triangle inequality)

◁

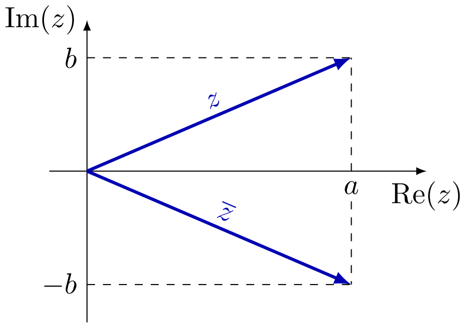


Example 4.6. For $z_1 = 1 + \mathrm{j}$ and $z_2 = 2 - 2\mathrm{j}$ we have

- $|z_1| = \sqrt{2} \geq 0 \quad$ and $\quad |z_2| = \sqrt{8} \geq 0$
- $|z_1 \cdot z_2| = 4 = \sqrt{2} \cdot \sqrt{8} = |z_1| \cdot |z_2|$
- $|z_1 + z_2| = \sqrt{10} \leq \sqrt{2} + \sqrt{8} = |z_1| + |z_2|$

◁

Definition 4.12 (Conjugation)**.** The *conjugate* of a complex number $z = a + \mathrm{j}\,b \in \mathbb{C}$ is defined by $\overline{z} = a - \mathrm{j}\,b$. A pair z and $\overline{z}$ is called *complex conjugates*. ◁



Theorem 4.13 (Calculating with conjugates)**.** With $z \in \mathbb{C}$ we have

- $\overline{\overline{z}} = z$
- $z \cdot \overline{z} = |z|^2, \quad$ i.e. $z \cdot \overline{z} \geq 0$ and $z \cdot \overline{z} \in \mathbb{R}$
- $z + \overline{z} = 2 \cdot \mathrm{Re}(z)$
- $z - \overline{z} = 2\mathrm{j} \cdot \mathrm{Im}(z)$

◁

Example 4.7. For $z = 2 - 3\mathrm{j}$ we have:

- $\overline{\overline{z}} = 2 - 3\mathrm{j}$
- $z \cdot \overline{z} = 13$
- $z + \overline{z} = 4$
- $z - \overline{z} = -6\mathrm{j}$

◁

Theorem 4.14 (Powers of complex numbers)**.** With $z = |z|\, e^{\mathrm{j}\,\varphi} \in \mathbb{C}$ and $n \in \mathbb{Z}$ we have

$$z^n = |z|^n(\cos n\varphi + \mathrm{j}\,\sin n\varphi) = |z|^n e^{\mathrm{j}n\varphi}$$

◁

Example 4.8. For $z_1 = 1 + \mathrm{j}$ and $z_2 = 2 - 2\mathrm{j}$ we have

$$
\begin{aligned}
{z_1}^2 &= 2\mathrm{j} & {z_2}^2 &= -8\mathrm{j} \\
{z_1}^3 &= -2 + 2\mathrm{j} & {z_2}^3 &= -16 - 16\mathrm{j} \\
{z_1}^4 &= -4 & {z_2}^4 &= -64 \\
{z_1}^{-2} &= -\frac{1}{2}\mathrm{j} & {z_2}^{-2} &= \frac{1}{8}\mathrm{j}
\end{aligned}
$$

◁

Definition 4.15 (n^{th}-root of complex numbers)**.** $z \in \mathbb{C}$ is called the n^{th}-root of $y \in \mathbb{C}$ if

$$z^n = y$$

$\triangleleft$

Remark: Care must be taken when working with roots in the context of complex numbers, e.g.

$$1 = \sqrt{1} = \sqrt{(-1)\cdot(-1)} \neq \sqrt{-1}\cdot\sqrt{-1} = \mathrm{j}\cdot\mathrm{j} = -1$$

The next theorem deals more general with roots for complex numbers.

Theorem 4.16 (Number of roots)**.** For $z \in \mathbb{C}$, $y = |y|e^{\mathrm{j}\,\varphi} \in \mathbb{C}_{\neq 0}$ and $n \in \mathbb{N}$ the equation

$$z^n = y$$

has exactly n solutions for z which are:

$$z_k = \sqrt[n]{|y|}\exp(\mathrm{j}\tfrac{\varphi+2\pi k}{n}) \quad \text{for } k = 0, 1, \ldots, n-1$$

$\triangleleft$

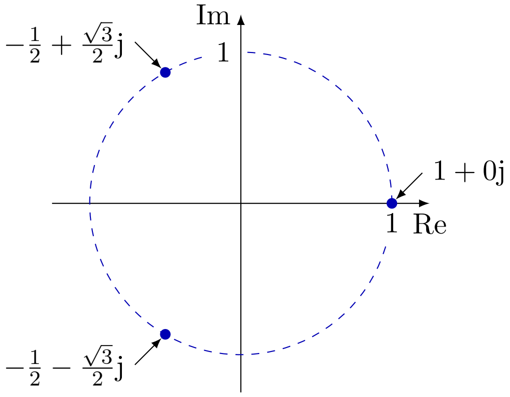
Example 4.9. n^{th} root of 1. From real numbers we know already that the square root ($n = 2$) has two solutions. With complex numbers we may write:

$$\begin{aligned} 1 &= (1+0\mathrm{j})^2 = \left(e^{\mathrm{j}\,0}\right)^2 \\ 1 &= (-1+0\mathrm{j})^2 = \left(e^{\mathrm{j}\,\pi}\right)^2 \end{aligned}$$

In exponential form we have the two angles $\{0, \pi\}$: Hence, the equation $z = \sqrt[2]{1}$ has the two solutions $\{1, -1\}$.

For cubic roots ($n = 3$) we have three solutions:

$$\begin{aligned} 1 &= \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}\mathrm{j}\right)^3 = \left(e^{-\mathrm{j}\,2\pi/3}\right)^3 \\ 1 &= (1+0\mathrm{j})^3 = \left(e^{-\mathrm{j}\,0}\right)^3 \\ 1 &= \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\mathrm{j}\right)^3 = \left(e^{\mathrm{j}\,2\pi/3}\right)^3 \end{aligned}$$



For arbitrary $n \in \mathbb{N}$ the roots are evenly distributed on the unit cycle on the complex plane. I.e. for $z = \sqrt[n]{1}$ we have

$$z_k = e^{\mathrm{j}2\pi k/n} \quad \text{for } k = 0, 1, \ldots, n-1$$

$\triangleleft$

Example 4.10. The equation

$$z^3 = 8$$

has the solutions

$$\begin{aligned} z_0 &= 2 \\ z_1 &= 2e^{\mathrm{j}\,2\pi/3} \\ z_2 &= 2e^{-\mathrm{j}\,2\pi/3} \end{aligned}$$

$\triangleleft$

4.5 Problems

Problem 4.1: For $n \in \mathbb{Z}$ simplify the following expressions:

1. j^2
2. j^5
3. j^{4n}
4. j^{4n+3}
5. j^{2n}

Problem 4.2: Solve the following equations:

1. $\mathrm{Re}(1 - \mathrm{j})$
2. $\mathrm{Im}(3 + 2\mathrm{j})$
3. $\mathrm{Re}(-a + b\mathrm{j})$
4. $\mathrm{Im}(\mathrm{j}^2)$

Problem 4.3: Check if the set of complex numbers as in definition 4.4 is a field.

Problem 4.4: Plot the following complex numbers in the complex plane:

1. $z_1 = 1 + 2\mathrm{j}$
2. $z_2 = 2 - \mathrm{j}$
3. $z_3 = -3 + 2\mathrm{j}$

4. $z_4 = -2 - 2\mathrm{j}$

Problem 4.5: With $z = a + \mathrm{j}\,b = |z| \cdot e^{\mathrm{j}\varphi}$ complete the following table:

a	b	$\lvert z\rvert$	φ
2	2		
-3	3		
		2	$\pi/2$
		4	$-\pi/4$

Problem 4.6: Plot the following complex numbers in the complex plane:

1. $z_1 = \sqrt{2}\, e^{\mathrm{j}\pi/4}$
2. $z_2 = \sqrt{8}\, e^{-\mathrm{j}3\pi/4}$
3. $z_3 = 2\, e^{-\mathrm{j}\pi/2}$
4. $z_4 = e^{\mathrm{j}\pi}$

Problem 4.7: For $z_1 = 2 + 3\mathrm{j}$ and $z_2 = 3 - 2\mathrm{j}$ solve the following expressions:

1. $z_1 + z_2$
2. $z_2 - z_1$
3. $z_1 \cdot z_2$
4. z_1/z_2
5. ${z_1}^3$
6. $(z_2/z_1)^2$

Problem 4.8: For $z_1 = 2\, e^{\mathrm{j}\pi/4}$ and $z_2 = e^{\mathrm{j}\pi/3}$ solve the following expressions:

1. $z_1 \cdot z_2$
2. ${z_1}^2$
3. ${z_2}^2$
4. ${z_1}^2 \cdot {z_2}^3$
5. $\mathrm{Re}({z_2}^2)$
6. $\mathrm{Im}({z_1}^2)$
7. z_2/z_1

Problem 4.9: For $z_1 = 2 + 3\mathrm{j}$ and $z_2 = 3 - 2\mathrm{j}$ solve the following expressions:

1. $z_1 + \overline{z_1}$
2. $\overline{z_2} - z_2$
3. $z_1 - \overline{z_2}$
4. $z_1 \cdot \overline{z_1} + z_2 \cdot \overline{z_2}$
5. $z_1 \cdot \overline{z_1} - |z_1| \cdot |z_2|$

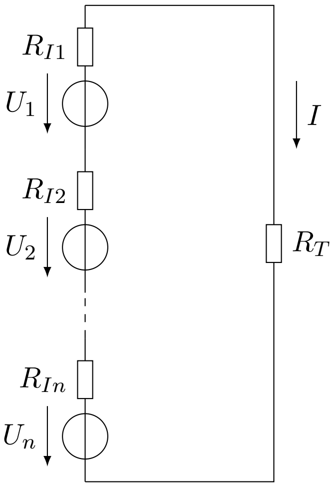
Problem 4.10: Solve the following equations for $z \in \mathbb{C}$:

1. $z = \sqrt[4]{1}$
2. $z = \sqrt[2]{-1}$
3. $z = \sqrt[4]{25}$
4. $z = \sqrt[4]{-324}$
5. $z = \sqrt[8]{81}$

5 Sequences

5.1 Introduction

Example 5.1. In electronics we deal with resistors and voltage supplies. Lets assume a terminating resistor R_L and a variable number of voltage supplies with equal voltage $U_1 = U_2 = \ldots = U$ and equal internal resistance $R_{i1} = R_{i2} = \ldots = R_i$. The voltage supplies are connected in a chain, i.e. the voltage and internal resistance increases with increasing number of voltage supplies.



What is the total current I through the terminating resistor R_L? For one voltage supply we get by Ohm's law:

$$I_1 = \frac{U}{R_I + R_T}$$

For two voltage supplies we get:

$$I_2 = \frac{U + U}{R_I + R_I + R_T} = \frac{2 \cdot U}{2 \cdot R_I + R_T}$$

For an arbitrary number n of voltage supplies we get:

$$I_n = \frac{n \cdot U}{n \cdot R_I + R_T} = \frac{U}{R_I + \frac{R_T}{n}}$$

To find the maximum possible electrical current we increase the number of voltage supplies towards infinity:

$$I_{\max} = \lim_{m \to \infty} \frac{U}{R_I + \frac{R_T}{n}} = \frac{U}{R_I}$$

Hence, for every number of voltage supplies we found an associated total current and a limiting value for an infinite number of voltage supplies.

Let's assume the voltage supplies have a voltage of 1 V with an internal resistor of 1 Ω. With a terminating resistor of 3 Ω we get for I in ampere:

$$I_1 = \tfrac{1}{4} \qquad I_2 = \tfrac{2}{5} \qquad I_3 = \tfrac{3}{6} \qquad I_4 = \tfrac{4}{7} \quad \ldots$$

Noted as a sequence we write in short:

$$(I_n) = (\tfrac{1}{4}, \tfrac{2}{5}, \tfrac{3}{6}, \tfrac{4}{7}, \tfrac{5}{8}, \tfrac{6}{9}, \tfrac{7}{10}, \tfrac{8}{11}, \ldots)$$

◁

5.2 Definition of sequences

Definition 5.1 (Sequence)**.** A *sequence* maps $\mathbb{N}$ to $\mathbb{R}$ or $\mathbb{C}$. We write $(x_n)_{n \in \mathbb{N}}$ or just (x_n) where every $n \in \mathbb{N}$ is mapped to a real or complex number x_n. ◁

Example 5.2. Some sequences:

- The sequence (a_n) with $a_n = \frac{1}{n!}$ is a sequence of real numbers:

  $$a_1 = \tfrac{1}{1!},\ a_2 = \tfrac{1}{2!},\ a_3 = \tfrac{1}{3!},\ a_4 = \tfrac{1}{4!},\ \ldots$$
  $$(a_n) = (\tfrac{1}{1!}, \tfrac{1}{2!}, \tfrac{1}{3!}, \tfrac{1}{4!}, \ldots)$$

- The sequence (b_n) with $b_n = 2$ is an example of a *constant sequence*:

  $$(b_n) = (2, 2, 2, 2, 2, \ldots)$$

- The sequence (c_n) with $c_n = \mathrm{j}^n$ is complex and periodic:

  $$(c_n) = (\mathrm{j}, -1, -\mathrm{j}, 1, \mathrm{j}, -1, -\mathrm{j}, 1, \ldots)$$

◁

Definition 5.2 (Bounded sequence)**.** A sequence (a_n) is said to be *bounded* if there exists a limit $M \in \mathbb{R}$ such that

$$|a_n| \leq M \qquad \text{for all } n \in \mathbb{N}$$

◁

Example 5.3.

- (a_n) with $a_n = \frac{1}{n}$ is bounded
- (b_n) with $b_n = n^2$ is not bounded
- (c_n) with $c_n = (-1)^n$ is bounded
- (d_n) with $d_n = x^n$, $x \in \mathbb{C}$ is bounded for $|x| \leq 1$

◁

Definition 5.3 (Monotonicity)**.** Let (a_n) be a sequence of real numbers. (a_n) is called

$$\begin{aligned} &\textit{increasing} && \text{if} && a_{n+1} \geq a_n \\ &\textit{decreasing} && \text{if} && a_{n+1} \leq a_n \\ &\textit{strictly increasing} && \text{if} && a_{n+1} > a_n \\ &\textit{strictly decreasing} && \text{if} && a_{n+1} < a_n \end{aligned}$$

for all $n \in \mathbb{N}$. If a sequence is (strictly) increasing or (strictly) decreasing we call it (strictly) *monotone.* ◁

Example 5.4.

- (a_n) with $a_n = \frac{1}{n}$ is strictly decreasing
- (b_n) with $b_n = n^2$ is strictly increasing
- (c_n) with $c_n = (-1)^n$ is not monotone
- (d_n) with $d_n = x^n$ is monotone for $x \in \mathbb{R}_{\geq 0}$ and strictly monotone for $x \in \mathbb{R}_{>0} \setminus \{1\}$

◁

5.3 Convergence and divergence

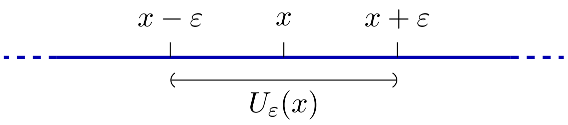
Definition 5.4 (Epsilon-neighbourhood)**.** Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. For a number $x \in \mathbb{K}$ and $\varepsilon \in \mathbb{R}_{>0}$ we say

$$U_\varepsilon(x) = \{y \in \mathbb{K} \mid |y - x| < \varepsilon\}$$

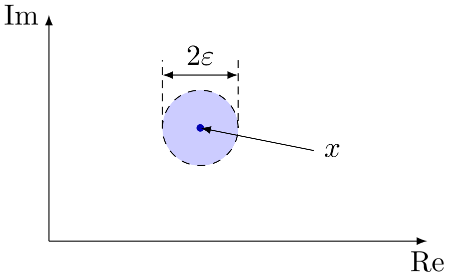
is the *epsilon-neighbourhood* of x. ◁

Remark: For real numbers the epsilon-neighbourhood equals the open interval between $x - \varepsilon$ and $x + \varepsilon$:

$$U_\varepsilon(x) = (x - \epsilon, x + \epsilon) \quad \text{for } x \in \mathbb{R}.$$



For complex numbers the epsilon-neighbourhood equals a circular disk on the complex plane with diameter 2ε. All numbers inside this disk (excluding the edge) belong to the epsilon-neighbourhood of x.



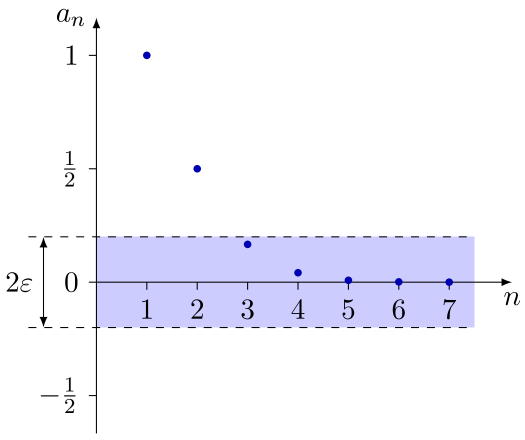
Definition 5.5 (Convergence)**.** Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. A sequence (a_n) with $a_n \in \mathbb{K}$ is said to be *convergent* with limiting value a if for any $\varepsilon \in \mathbb{R}_{>0}$ there exist an $N \in \mathbb{N}$ such that all a_n with $n \geq N$ are elements of the epsilon-neighbourhood $U_\varepsilon(a)$. In short:

$$\forall \varepsilon \in \mathbb{R}_{>0}\ \exists N \in \mathbb{N}\ \forall n \geq N : |a_n - a| < \varepsilon$$

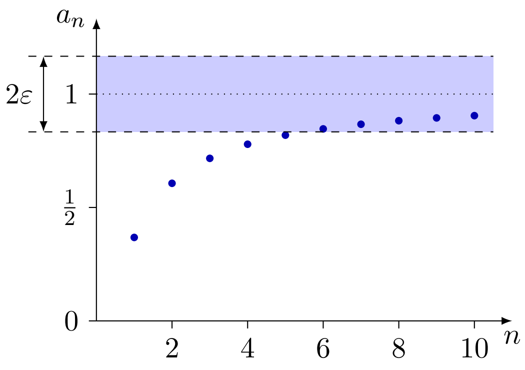
◁

Example 5.5.

- The sequence (a_n) with $a_n = \frac{1}{n!}$ is convergent.



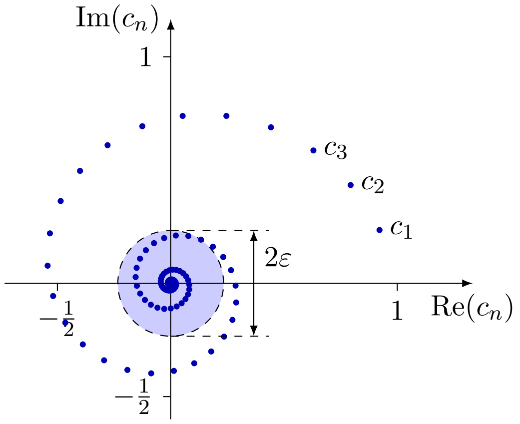
- The sequence (b_n) with $b_n = e^{-1/n}$ is convergent.



- The sequence (c_n) with

$$\begin{aligned} c_n &= e^{-n/20+\mathrm{j}\,n/4} \\ &= e^{-n/20}[\cos(\tfrac{n}{4}) + \mathrm{j}\,\sin(\tfrac{n}{4})] \end{aligned}$$

  is convergent.



◁

Remark: A sequence is said to be a *zero sequence* if its elements converge towards zero. E.g. in the previous example (a_n) and (c_n) are zero sequences.

Definition 5.6 (Divergence)**.** Non-convergent sequences are called *divergent* sequences. ◁

Example 5.6.

- (a_n) with $a_n = n$ is not convergent, i.e. divergent.
- (b_n) with $b_n = \mathrm{j}^n$ is divergent.
- (c_n) with $c_n = x^n$, $x \in \mathbb{R}$ is convergent for $x \in (-1, 1]$ and divergent otherwise.
- (d_n) with $d_n = z^n$, $z \in \mathbb{C}$ is convergent for $|z| < 1 \wedge z = 1$ and divergent otherwise.

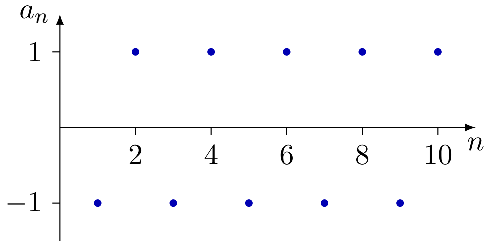
◁

Theorem 5.7 (Convergent sequences bounded)**.** Convergent sequences are bounded. ◁

Remark: This theorem states an *implication*: *If* a sequence is convergent *then* it is bounded too.

The reverse is not valid: A bounded sequence may or may not be convergent.

Example: The sequence (a_n) with $a_n = (-1)^n$ is bounded but not convergent.



Theorem 5.8 (Monotone bounded sequences)**.** Monotone bounded sequences are convergent. ◁

5.4 Limit of sequences

Definition 5.9 (Limit of a sequence)**.** For a convergent sequence (a_n) where the elements a_n converge towards a we write:

$$\lim_{n\to\infty} a_n = a$$

◁

Remark: Other (abbreviated) notations are:

$$\begin{aligned} \lim a_n &= a \\ a_n &\xrightarrow[n\to\infty]{} a \\ a_n &\longrightarrow a \end{aligned}$$

Example 5.7.

- for (a_n) with $a_n = \frac{1}{n}$ we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

- for (b_n) with $b_n = \frac{n}{n+1}$ we have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

- for (c_n) with $c_n = \frac{n+1}{n}$ we have

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} = 1$$

- for (d_n) with $d_n = \frac{x^n}{n!}$, $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{x^n}{n!} = 0$$

  since for $n > |x|$ the denominator increases faster than the numerator, i.e.

$$|d_{n+1}| < |d_n| \quad \text{for} \quad n > |x|$$

- for (e_n) with $e_n = e^{1/n}$ we have

$$\lim_{n\to\infty} e_n = \lim_{n\to\infty} e^{1/n} = e^0 = 1$$

◁

Remark: Obviously, for a zero sequence (a_n) we have:

$$\lim_{n\to\infty} a_n = 0$$

Hence, (a_n) and (d_n) in the previous example are zero sequences.

Definition 5.10 (Definite divergence)**.** A sequence (a_n) with $a_n \in \mathbb{R}$ is divergent towards $+\infty$ if for any $K \in \mathbb{R}$ there exist an $N \in \mathbb{N}$ such that for all a_n with $n \geq N$ we have $a_n > K$. In short:

$$\forall K \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n \geq N : a_n > K$$

A sequence (a_n) with $a_n \in \mathbb{R}$ is divergent towards $-\infty$ if for any $K \in \mathbb{R}$ there exist an $N \in \mathbb{N}$ such that for all a_n with $n \geq N$ we have $a_n < K$. In short:

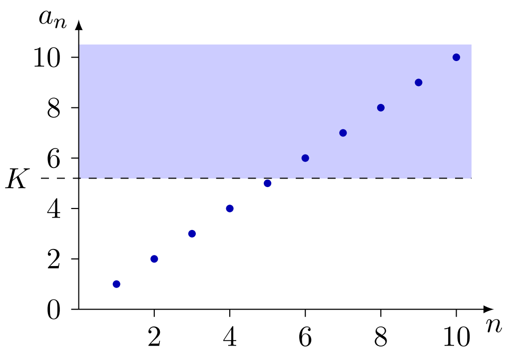
$$\forall K \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n \geq N : a_n < K$$

◁

Example 5.8.

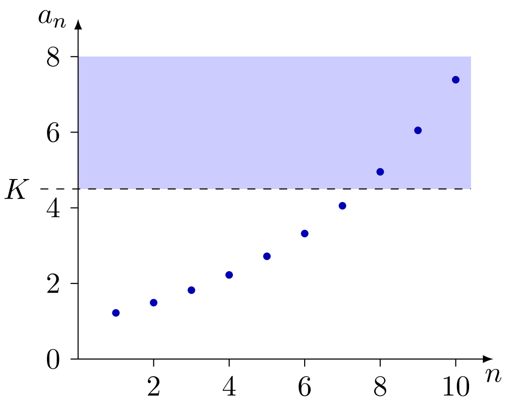
- For (a_n) with $a_n = n$ we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$$



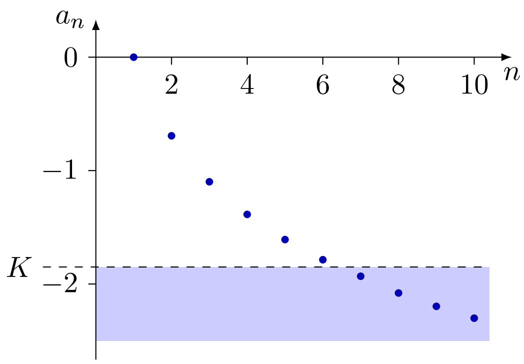
- For (b_n) with $b_n = \exp\left(\frac{n}{5}\right)$ we have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \exp\left(\frac{n}{5}\right) = \infty$$



- For (c_n) with $b_n = \ln(1/n)$ we have

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \ln\left(\frac{1}{n}\right) = -\infty$$



◁

Theorem 5.11 (Calculating with convergent sequences)**.** Let (a_n) and (b_n) be convergent sequences. We have for the sequence of sums $(a_n + b_n)$ and the sequence of products $(a_n \cdot b_n)$:

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} (a_n) + \lim_{n\to\infty} (b_n)$$
$$\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} (a_n) \cdot \lim_{n\to\infty} (b_n)$$

If $b_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} (b_n) \neq 0$ then we have for the sequence of quotients $(\frac{a_n}{b_n})$:

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} (a_n)}{\lim_{n\to\infty} (b_n)}$$

◁

Example 5.9. Let (a_n) be a sequence with $a_n = \frac{2n-1}{n}$ and (b_n) be a sequence with $b_n = \cos(\frac{1}{n})$. Then we have:

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n-1}{n} = \lim_{n\to\infty} \frac{2-\frac{1}{n}}{1} = 2$

- $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1$
- $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = 3$
- $\lim\limits_{n\to\infty} (a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n = 1$
- $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n = 2$
- $\lim\limits_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = 2$

◁

5.5 Problems

Problem 5.1: Evaluate the first five elements of the following sequences:

- (a_n) with $a_n = 3n$
- (b_n) with $b_n = \frac{n+1}{2n-1}$
- (c_n) with $c_n = \frac{n^2+n}{2}$
- (d_n) with $d_n = \sum_{k=1}^{n} k$
- (e_n) with $e_n = \mathrm{j}^{2n}$
- (f_n) with $f_n = \mathrm{j}^{-1}$

Problem 5.2: Which of the series in problem 5.5 are bounded?

Problem 5.3: Fill the following table with $\surd$ for the appropriate properties of the sequences in problem 5.5.

sequence	increasing	strictly increasing	decreasing	strictly decreasing	monotone	strictly monotone
(a_n)						
(b_n)						
(c_n)						
(d_n)						
(e_n)						
(f_n)						

Problem 5.4: Describe the the following expression for $\varepsilon = \frac{1}{2}$ for real numbers:

$$U_\varepsilon(2)$$

Problem 5.5: Describe the the following expression for $\varepsilon = \frac{1}{2}$ for complex numbers:

$$U_\varepsilon(\mathrm{j})$$

Problem 5.6: Describe *convergence* in your own words.

Problem 5.7: Which of the sequences in problem 5.5 are convergent, which of them are divergent?

Problem 5.8: Find the limit of the following sequences:

- (a_n) with $a_n = \frac{1}{n^2} + 1$
- (b_n) with $b_n = \frac{n+1}{2n-1}$
- (c_n) with $c_n = \frac{n^2-1}{2n^2}$
- (d_n) with $d_n = \frac{n^2}{1-2n}$
- (e_n) with $e_n = \tan\left(\frac{\pi n}{4n+1}\right)$

Problem 5.9: For the sequences in the previous problem find the limits of the following expressions:

- $(a_n + b_n)$
- $(a_n \cdot c_n)$
- $(c_n \cdot d_n)$
- $\left(\frac{e_n}{b_n}\right)$

6 Series

6.1 Introduction

Example 6.1. Achilles and the tortoise.

For some reasons Achilles, the famous runner of the Greek mythology, has a race with a tortoise. Obviously the tortoise has not the slightest chance and, hence, is given a head start of 100m. Let us assume that Achilles runs (only) twice as fast as the tortoise and both run with a constant speed.

Some time after the start Achilles reaches the 100m point where the tortoise started. However, the tortoise is not there but ran already 50m and is at 150m. So, Achilles runs another 50m, but again does not reach the tortoise who ran another 25m. Will Achilles ever reach the tortoise? And if yes, where will it be?

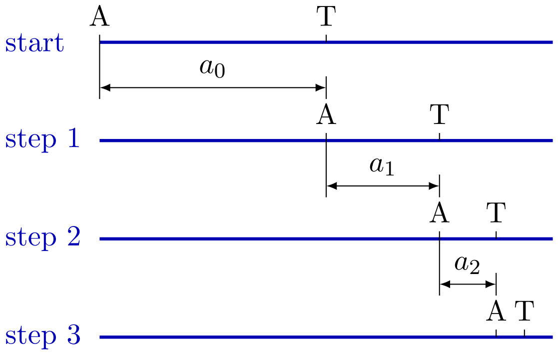
Of course, we could set up an equation and solve it for the point where the two runners meet. However, let us look at it as a sequence of distance steps of Achilles in meter: $a_0 = 100$, $a_1 = 50$, $a_2 = 25$, $a_3 = 12.5$ etc. So, we define a sequence by

$$(a_n) = (100, 50, 25, 12\tfrac{1}{2}, 6\tfrac{1}{4}, \ldots)$$

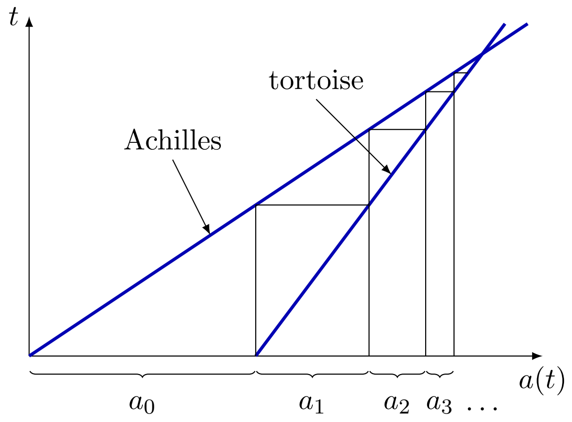
where

$$a_n = 100 \left(\frac{1}{2}\right)^n \qquad \text{for } n = 0, 1, 2, \ldots$$

The following sketch shows the first three distance steps of Achilles:



We also may look at it on a diagram where we plot the distance of Achilles and the tortoise on the abscissa and the time on the ordinate:

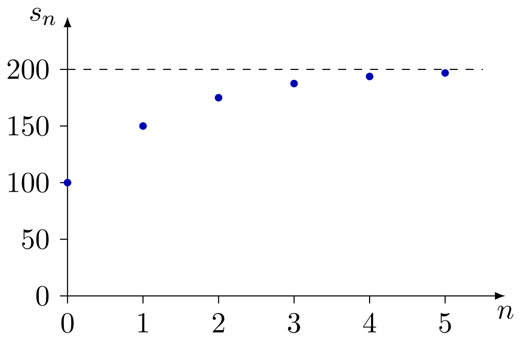


To find the point where Achilles and the tortoise meet we sum up the distance steps which gives us a sequence of partial sums. With

$$s_n = \sum_{k=0}^{n} a_k = 100 \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$$

we have

$$(s_n) = (100, 150, 175, 187\tfrac{1}{2}, 193\tfrac{3}{4}, \ldots)$$



From example 2.25 we know

$$\sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q}$$

hence with $q = \frac{1}{2}$ we get for the partial sums:

$$s_n = 100 \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = 200 \left(1 - \left(\tfrac{1}{2}\right)^{n+1}\right)$$

To get the point, where Achilles and the tortoise meet we have to find s_n for an infinite n

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 200 \left(1 - \underbrace{\left(\tfrac{1}{2}\right)^{n+1}}_{\to 0}\right) = 200$$

At last, they will meet at the 200m point! ◁

6.2 Definition of series

Definition 6.1 (Partial sum)**.** Let (a_n) be a sequence. The n^{th} *partial sum* s_n is defined by:

$$s_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \ldots + a_n$$

◁

Remark: If we combine all partial sums to a sequence (s_n) we call it the *sequence of partial sums.*

In the previous chapter we defined that a sequence maps $\mathbb{N}$ to $\mathbb{R}$ or $\mathbb{C}$, i.e. the elements of a sequence (a_n) with the subscripts $1, 2, 3, \ldots$. In this chapter we untighten this definition and allow the sequence to start at other values than one, e.g. at zero or two.

Example 6.2.

- For the Gaussian sum we have the sequence of natural numbers

  $$(a_n) = (1, 2, 3, 4, 5, 6, 7, \ldots)$$

  and the partial sums

  $$s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

  which gives us the sequence of partial sums:

  $$(s_n) = (1, 3, 6, 10, 15, 21, 28, \ldots)$$

- For the square numbers we have the sequence of odd numbers

  $$(a_n) = (1, 3, 5, 7, 9, 11, \ldots)$$

  and the partial sums

  $$s_n = \sum_{k=1}^{n} 2k - 1 = n^2$$

  which gives us the sequence of partial sums, i.e. in this case the sequence of square numbers:

  $$(s_n) = (1, 4, 9, 16, 25, 36, 49, \ldots)$$

- For the sequence

  $$(a_n) = \left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{32}, \ldots\right)$$

  we get the partial sums

  $$s_n = \sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

  which gives us the sequence of partial sums:

  $$(s_n) = \left(\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{7}{8}, \tfrac{15}{16}, \tfrac{31}{32}, \ldots\right)$$

◁

Definition 6.2 (Series)**.** Let (a_n) be a sequence. We call

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + a_3 + \ldots$$

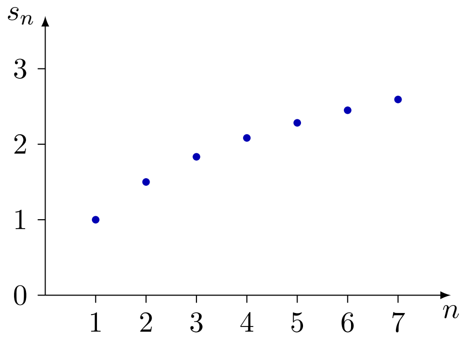
an (infinite) *series.* ◁

Remark: We call a_n is the n^{th} *series element.* If a sequence (a_n) starts with an index greater than zero the first few series elements may be treated as zero.

Example 6.3.

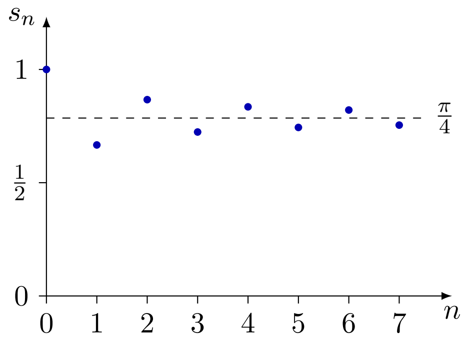
- Harmonic series:

  $$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$



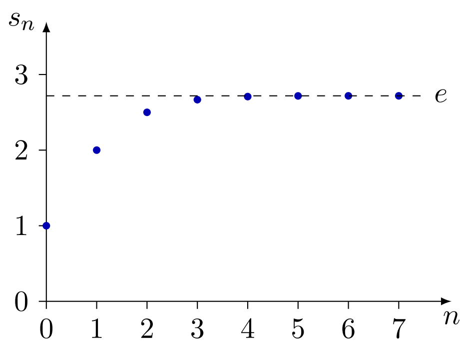
- Leibniz series:

  $$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$



- Euler's number:

$$\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$
$$= 2.71828\dots$$



◁

6.3 Convergence of series

Definition 6.3 (Convergence and divergence)**.** Let (a_n) be a sequence and (s_n) the corresponding sequence of partial sums, i.e. $s_n = \sum_{k=0}^{n} a_k$. If the sequence (s_n) is *convergent* we say the series $\sum_{k=0}^{\infty} a_k$ is convergent and the limit is given by:

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} s_n$$

Series that are not convergent are called *divergent.* ◁

Example 6.4.

- The series $\sum_{k=0}^{\infty} a_k$ with $a_k = \frac{1}{k!}$ is convergent with Euler's number as limiting value:

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{1}{k!} = e = 2.71828\dots$$

- The *harmonic series* $\sum_{k=0}^{\infty} b_k$ with $b_k = \frac{1}{k}$ is divergent:

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

This becomes clear when looking at the partial sums s_{2^n}, $n = 0, 1, 2, \dots$, i.e. s_1, s_2, s_4, s_8, s_{16} etc. The difference between these partial sums is $\geq \frac{1}{2}$.

$$1 + \underbrace{\tfrac{1}{2}}_{\geq\frac{1}{2}} + \underbrace{\tfrac{1}{3} + \tfrac{1}{4}}_{\geq\frac{1}{2}} + \underbrace{\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}}_{\geq\frac{1}{2}} + \dots$$

Hence we have

$$s_{2^n} \geq 1 + \tfrac{n}{2} \qquad \text{for } n = 1, 2, 3, \dots$$

Since the right term has a limit of ∞ the harmonic series diverges towards ∞ too.

- The series $\sum_{k=1}^{\infty} c_k$ with $c_k = \frac{(-1)^k}{k}$ is convergent:

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} = \ln(2)$$

◁

Theorem 6.4 (Calculating with series)**.** Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be convergent series and $c \in \mathbb{R}$ or $c \in \mathbb{C}$. Then the series $\sum_{k=0}^{\infty} (a_k + b_k)$ and $\sum_{k=0}^{\infty} c\, a_k$ are convergent too and we have:

$$\sum_{k=0}^{\infty} (a_k + b_k) = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$$
$$\sum_{k=0}^{\infty} c\, a_k = c \sum_{k=0}^{\infty} a_k$$

◁

Example 6.5. With the two sequences (a_n), $a_n = \frac{1}{2^n}$ and (b_n), $b_n = (-1)^n \frac{1}{2^n}$ both for $n = 0, 1, 2, \dots$ and $c = 3 + \mathrm{j}$ we have:

- $\sum_{k=0}^{\infty} a_k = 2$
- $\sum_{k=0}^{\infty} b_k = \frac{2}{3}$
- $\sum_{k=0}^{\infty} (a_k + b_k) = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k = 2\frac{2}{3}$
- $\sum_{k=0}^{\infty} c\, a_k = c \sum_{k=0}^{\infty} a_k = 6 + 2\mathrm{j}$
- $\sum_{k=0}^{\infty} c\, b_k = c \sum_{k=0}^{\infty} b_k = 2 + \frac{2}{3}\mathrm{j}$

◁

Example 6.6. Let (a_n) be a sequence with

$$a_n = \frac{1}{2^{n+1} + 2^n \mathrm{j}}$$

What is the limiting value for the sum of all elements of (a_n)?

$$\begin{aligned}\sum_{k=0}^{\infty} a_k &= \sum_{k=0}^{\infty} \frac{1}{2^{k+1}+2^k\mathrm{j}} = \sum_{k=0}^{\infty} \frac{1}{2\cdot 2^k+2^k\mathrm{j}}\\ &= \sum_{k=0}^{\infty} \frac{1}{(2+\mathrm{j})\cdot 2^k} = \frac{1}{2+\mathrm{j}}\sum_{k=0}^{\infty}\frac{1}{2^k}\\ &= \frac{1}{2+\mathrm{j}}\cdot 2 = \frac{2}{2+\mathrm{j}}\cdot\frac{2-\mathrm{j}}{2-\mathrm{j}} = \frac{4-2\mathrm{j}}{4+1}\\ &= \tfrac{4}{5}-\tfrac{2}{5}\mathrm{j}\end{aligned}$$

◁

6.4 Geometric series

Theorem 6.5 (Geometric sum)**.** We call

$$\sum_{k=0}^{n} x^k = 1+x+x^2+x^3+\ldots+x^n$$

geometric sum. For any $x \in \mathbb{C}_{\neq 1}$ we have:

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

◁

Remark: The term *geometric sum* comes from the fact that each summand is the *geometric mean* of the two neighbour summands:

$$a_k = \sqrt{a_{k-1}a_{k+1}}$$

i.e.

$$a_k = \sqrt{x^{k-1}x^{k+1}} = \sqrt{x^{2k}} = x^k$$

Example 6.7.

- $\displaystyle\sum_{k=0}^{7} 3^k = \frac{1-3^8}{1-3} = \frac{-6560}{-2} = 3280$
- $\displaystyle\sum_{k=0}^{4} \frac{1}{2^k} = \sum_{k=0}^{4}\left(\frac{1}{2}\right)^k = \frac{1-\left(\frac{1}{2}\right)^5}{1-\frac{1}{2}} = 1\tfrac{15}{16}$
- $\displaystyle\sum_{k=0}^{3} \frac{1}{10^k} = \frac{1-\left(\frac{1}{10}\right)^4}{1-\frac{1}{10}} = \tfrac{1111}{1000} = 1.111$

◁

Example 6.8. In computer science unsigned binary integers consist of a number of binary digits each with the two possible values zero and one. The rightmost digit (bit 0) is weighted with a factor 1 (i.e. 2^0), the second digit from the right (bit 1) has the factor 2 (i.e. 2^1), the third (bit 2) the factor 4 (i.e. 2^2) and so on.

What is the greatest number to be stored in unsigned integers with n bits? We notice that we deal with the geometric sum and that the leftmost bit has the number $n-1$.

$$\sum_{k=0}^{n-1} 2^k = \frac{1-2^n}{1-2} = 2^n - 1$$

Some typical examples:

- 1 byte = 8 bit: 255
- 2 byte = 16 bit: 65 535
- 4 byte = 32 bit: 4 294 967 295

◁

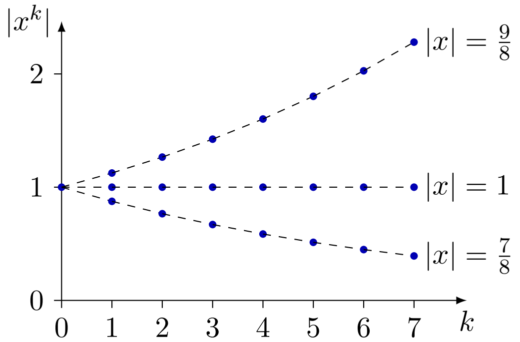
Theorem 6.6 (Geometric series)**.** The *geometric series*

$$\sum_{k=0}^{\infty} x^k = 1+x+x^2+x^3+\ldots$$

with $x\in\mathbb{C}$ is convergent for $|x|<1$ and we have:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

◁



Example 6.9. We come back to example 6.1 in the introduction of this chapter. The we found the sequence (a_n) with $a_n = 100\left(\frac{1}{2}\right)^n$ for $n = 0,1,2,\ldots$. To find the meeting point of Achilles

and the tortoise we take the infinite geometric sum:

$$\sum_{k=0}^{\infty} a_k = 100 \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{100}{1-\frac{1}{2}} = 200$$

◁

Example 6.10. What is 0.999 999 . . .? Each digit behind the dot has a weighting factor of $\frac{1}{10}$ of its left neighbour. Hence, we may write:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{9}{10^k} &= \sum_{k=0}^{\infty} \frac{9}{10^k} - 9 = 9 \sum_{k=0}^{\infty} \left(\frac{1}{10}\right)^k - 9 \\ &= 9 \frac{1}{1-\frac{1}{10}} - 9 = 1 \end{aligned}$$

◁

Remark: The fact that the geometric series is convergent for $|x| < 1$ serves as a tool to prove convergence of arbitrary series, see theorem 6.11 and 6.12. There we use two properties of the convergent geometric series:

$$\begin{aligned} \sqrt[n]{|a_n|} &= \sqrt[n]{|x^n|} = |x| < 1 \\ \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{x^{n+1}}{x^n}\right| = |x| < 1 \end{aligned}$$

6.5 Convergence criteria

Theorem 6.7 (Convergent series)**.** If a series $\sum_{k=0}^{\infty} a_k$ is convergent, then the sequence (a_n) is a zero sequence. ◁

Remark: This is an implication and, hence, is not valid in the other direction. However, due to this theorem a necessary but not sufficient constraint for convergence of a series is that the corresponding sequence (a_n) is a zero sequence.

Example 6.11.

- For all examples with convergent series $\sum_{k=0}^{\infty} a_k$ in this chapter the sequences (a_n) are zero sequences.
- The *harmonic series* $\sum_{k=1}^{\infty} a_k$ with $a_k = \frac{1}{k}$ is divergent, however, the sequence (a_n) is a zero sequence.

◁

Definition 6.8 (Absolute convergence)**.** Let $\sum_{k=0}^{\infty} a_k$ be a series. If the series $\sum_{k=0}^{\infty} |a_k|$ is convergent, then we call $\sum_{k=0}^{\infty} a_k$ *absolute convergent.* ◁

Theorem 6.9 (Absolute convergent series are convergent)**.** Every absolute convergent series is convergent. ◁

Remark: If we find that a series is convergent for the absolute of its elements it is convergent for any signs of the elements. However, if a series is absolute divergent it may or may not be convergent.

Example 6.12.

- The series $\sum_{k=1}^{\infty} a_k$ with $a_k = \pm\frac{1}{k^2}$ is absolute convergent and, hence, is convergent for any signs of a_k.
- The series $\sum_{k=1}^{\infty} a_k$ with $a_k = \frac{(-1)^{k+1}}{k}$ is absolute divergent (harmonic series) but still convergent with the limit $\ln 2$.

◁

Theorem 6.10 (Comparison test)**.** If the series $\sum_{k=0}^{\infty} b_k$ is absolute convergent, then the series $\sum_{k=0}^{\infty} a_k$ is absolute convergent too if for almost all summands we have $|a_k| \leq |b_k|$.

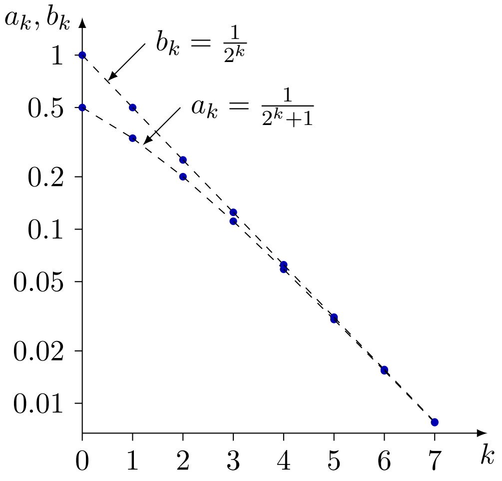
Let $\sum_{k=0}^{\infty} b_k$ be an absolute divergent series. The series $\sum_{k=0}^{\infty} a_k$ is absolute divergent too if for almost all summands we have $|a_k| \geq |b_k|$. (Although being absolute divergent it still may be convergent for appropriate signs of the summands.) ◁

Example 6.13. Is the series $\sum_{k=0}^{\infty} a_k$ with $a_k = \frac{1}{2^k+1}$ convergent? We look for convergent series that serves as a limit for the comparison test.

The geometric series $\sum_{k=0}^{\infty} b_k$ with $b_k = \frac{1}{2^k}$ is convergent and we have:

$$|a_k| \leq |b_k| \qquad \text{for all } k \in \mathbb{N}$$

Hence, the series $\sum_{k=0}^{\infty} a_k$ in convergent.



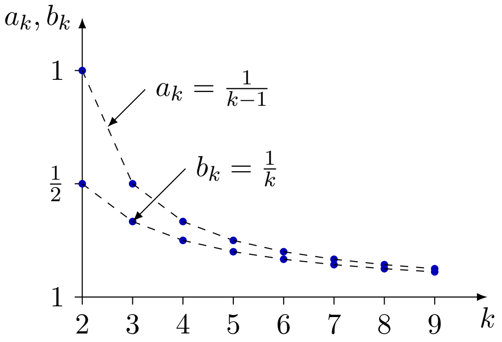
◁

Example 6.14. Is the series $\sum_{k=2}^{\infty} a_k$ with $a_k = \frac{1}{k-1}$ convergent? We look for convergent series that serves as a limit for the comparison test.

The harmonic series $\sum_{k=1}^{\infty} b_k$ with $b_k = \frac{1}{k}$ is divergent and we have:

$$|a_k| \geq |b_k| \qquad \text{for all } k \in \mathbb{N}, k \geq 2$$

Hence, the series $\sum_{k=0}^{\infty} a_k$ is absolute divergent.



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Theorem 6.11 (Root test)**.** Let $\sum_{k=0}^{n} a_k$ be a series. If there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\sqrt[n]{|a_n|} < 1$$

the series is absolute convergent. If there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\sqrt[n]{|a_n|} > 1$$

the series is divergent. I.e. for

$$q = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

if $q < 1$ the series is absolute convergent, if $q > 1$ the series is divergent and if $q = 1$ the root test does not give evidence on convergence. ◁

Remark: The root test has been developed by Augustin-Louis Cauchy and, hence, is sometimes called *Cauchy root test* or *Cauchy's radical test.*

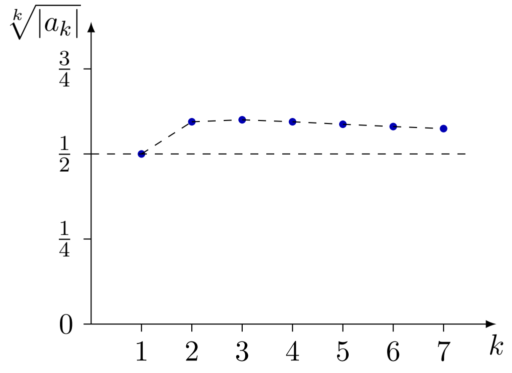
Example 6.15. Is the series $\sum_{k=0}^{\infty} a_k$ with

$$a_k = \frac{\sqrt{k}}{2^k}$$

convergent? We apply the root test:

$$\begin{aligned} q &= \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{\sqrt{n}}{2^n}\right|} \\ &= \lim_{n\to\infty} \sqrt[n]{\sqrt{n}\cdot\left(\tfrac{1}{2}\right)^n} = \lim_{n\to\infty} \sqrt[n]{\sqrt{n}}\cdot\sqrt[n]{\left(\tfrac{1}{2}\right)^n} \\ &= \lim_{n\to\infty} \underbrace{\sqrt[2n]{n}}_{\to 1}\cdot\tfrac{1}{2} = \tfrac{1}{2} \end{aligned}$$

Hence, the series is convergent.



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Theorem 6.12 (Ratio test)**.** Let $\sum_{k=0}^{n} a_k$ be a series. If there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| < 1$$

the series is absolute convergent. If there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| > 1$$

the series is divergent. I.e. for

$$q = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

if $q < 1$ the series is absolute convergent, if $q > 1$ the series is divergent and if $q = 1$ the ratio test does not give evidence on convergence. ◁

Example 6.16. Does the series $\sum_{k=0}^{\infty} a_k$ with $a_k = z^{2n}$, $z \in \mathbb{C}$ converge? We apply the ratio test:

$$\begin{aligned} q = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{z^{2n+2}}{z^{2n}}\right| \\ &= \lim_{n\to\infty} |z^2| = |z^2| = z \cdot \overline{z} \end{aligned}$$

The series is convergent for $|z| < 1$ and divergent for $|z| > 1$. For $|z| = 1$ the ratio test does not give evidence on convergence. ◁

Theorem 6.13 (Alternating series test)**.** Let (a_n) be a zero sequence with positive elements. Then the series

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

is convergent. ◁

Remark: The alternating series test has been discovered by Gottfried Leibniz and, hence, is sometimes called *Leibniz test* or *Leibniz criterion*.

Example 6.17. The *harmonic series*

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

is divergent. However, since the series elements form a zero sequence, a series with the same elements but alternating signs is convergent:

$$\sum_{k=1}^{\infty} \frac{(-1)^{l+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2$$

◁

6.6 Problems

Problem 6.1: Solve the following expressions:

$$a = \sum_{k=1}^{5} k \qquad b = \sum_{k=1}^{4} \frac{1}{k}$$

$$c = \sum_{k=4}^{7} \frac{1}{k-3} \qquad d = \sum_{k=1}^{5} (2k-1)$$

Problem 6.2: Evaluate for the following sequences the first 7 elements ($n = 0, 1, \dots, 6$) of the sequence of partial sums (s_n):

1. (a_n) with $a_n = 1$ for $n = 0, 1, 2, \dots$
2. (b_n) with $b_n = 2n - 1$ for $n = 1, 2, 3, \dots$
3. (c_n) with $c_n = \frac{1}{2^{n-2}}$ for $n = 0, 1, 2, \dots$
4. (d_n) with $d_n = n^2$ for $n = 0, 1, 2, \dots$

Problem 6.3: Plot the first 7 elements of the following series in a diagram:

1. $\sum_{k=0}^{\infty} 2^{-k}$
2. $\sum_{k=1}^{\infty} 3^{-k}$
3. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$
4. $\sum_{k=0}^{\infty} \frac{1}{k+1}$

Problem 6.4: With the two known series

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e \qquad \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = \ln 2$$

find the limit of the following expressions:

$$a = \sum_{k=0}^{\infty} \frac{-1}{k!} \qquad b = \sum_{k=0}^{\infty} \frac{(-1)^k}{3k+3}$$

$$c = \sum_{k=0}^{\infty} \left(\frac{2}{k!} + \frac{(-1)^k}{k+1}\right) \qquad d = \sum_{k=0}^{\infty} \frac{2\mathrm{j} \cdot (-1)^k}{k+1}$$

Problem 6.5: Solve the following terms:

$$a = \sum_{k=0}^{11} 2^k \qquad b = \sum_{k=0}^{11} 2^{-k}$$

$$c = \sum_{k=0}^{5} 4^k \qquad d = \sum_{k=0}^{9} 1^k$$

$$e = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k \qquad f = \sum_{k=0}^{\infty} 3^{-k}$$

Problem 6.6: Check the following series for convergence by comparison test:

1. $\sum_{k=0}^{\infty} \frac{1}{k-1}$
2. $\sum_{k=0}^{\infty} \frac{3^k}{4^k+1}$
3. $\sum_{k=0}^{\infty} \frac{1}{\sqrt{k^2-1}}$
4. $\sum_{k=0}^{\infty} \frac{k}{k^3+k^2-3}$

Problem 6.7: Check the following series for convergence by root test:

1. $\displaystyle\sum_{k=0}^{\infty} \frac{3^k}{k^k}$
2. $\displaystyle\sum_{k=0}^{\infty} \frac{k^k}{5^{2k}}$
3. $\displaystyle\sum_{k=0}^{\infty} \left(\frac{k^2+3k}{1-4k^2}\right)^k$
4. $\displaystyle\sum_{k=0}^{\infty} \frac{k}{(-3)^k}$

Problem 6.8: Check the following series for convergence by ratio test:

1. $\displaystyle\sum_{k=0}^{\infty} \frac{5^k}{k!}$
2. $\displaystyle\sum_{k=0}^{\infty} \frac{4^k}{2^k+3^k}$
3. $\displaystyle\sum_{k=0}^{\infty} k\, e^{-k}$
4. $\displaystyle\sum_{k=0}^{\infty} k^3\, e^{-k}$

Problem 6.9: Find out which statements are true:

1. If a sequence (a_n) is a zero sequence, then the series $\sum_{k=0}^{\infty} a_n$ is convergent.
2. If a series is absolute convergent, then it is convergent.
3. If a sequence of partial sums is convergent, then the corresponding series is convergent.
4. If a series $\sum_{k=0}^{\infty} a_n$ is convergent, then the sequence (a_n) is a zero sequence.
5. If a series is convergent, then it is absolute convergent.
6. If a series is convergent, then its sequence of partial sums is convergent.

7 Power-series

7.1 Definition of power series

Definition 7.1 (Power series)**.** Let (a_n) be a sequence, $z_0 \in \mathbb{C}$ fixed and $z \in \mathbb{C}$. We define a *power series* with

$$\sum_{k=0}^{\infty} a_k z^k$$

or

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

and call z_0 *expansion point.* ◁

Remark: The expansion point $z_0 \in \mathbb{C}$ acts like an offset for z. It is sufficient to study power series without this offset knowing that we can apply the gained knowledge to the more general power series by shifting its domain.

Hence, from now on we concentrate on power series with expansion point 0.

Example 7.1. Some power series (the first is a 2^{nd} order polynomial):

- $f_1(z) = 1 + 3z - 2z^2 + 0z^3 + 0z^4 + \dots$
- $f_2(z) = \displaystyle\sum_{k=0}^{\infty} z^k = 1 + z + z^2 + z^3 + \dots$
- $f_3(z) = \displaystyle\sum_{k=0}^{\infty} k! z^k = 1 + z + 2!z^2 + 3!z^3 + \dots$
- $f_4(z) = \displaystyle\sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = e^z$
- $f_5(z) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} z^k = 1 - \frac{z}{2} + \frac{z^2}{3} - \dots$
- $f_6(z) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} z^k = \frac{1}{2} - \frac{z}{4} + \frac{z^2}{8} - \dots$

◁

7.2 Convergence of power series

Definition 7.2 (Convergence)**.** If a power series has a limit we call it *convergent* – and *divergent* otherwise. For all $z \in \mathbb{C}$ where the power series is convergent we may treat the power series as a function and write:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

◁

Remark: How do we find out whether a power series is convergent? With

$$\sum_{k=0}^{\infty} a_k z^k$$

we apply the ratio test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1} z^{n+1}}{a_n z^n} \right| &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} z \right| \\ &= |z| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= |z| q < 1 \end{aligned}$$

where q is the ratio to check convergence of the series $\sum_{k=0}^{\infty} a_k$:

$$q = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

If $q = 0$ then the power series is absolute convergent for any $z \in \mathbb{C}$. If $q = \infty$ the power series is convergent for $z = 0$ only. If $0 < q < \infty$ then the power series is:

absolute convergent	for	$\lvert z\rvert < \frac{1}{q}$
divergent	for	$\lvert z\rvert > \frac{1}{q}$
unclear	for	$\lvert z\rvert = \frac{1}{q}$

(The same may be done with the root test.)

This leads us to the *main theorem of power series*:

Theorem 7.3 (Main theorem on power series)**.** Let (a_n) be a sequence and $z \in \mathbb{C}$. For the power series

$$\sum_{k=0}^{\infty} a_k z^k$$

one of the following holds:

1. convergence for $z = 0$ only
2. convergence for all $z \in \mathbb{C}$
3. There is a number $r \in \mathbb{R}$, $r > 0$ such that the power series is absolute convergent for all $|z| < r$ and divergent for all $|z| > r$.

This number r is called *radius of convergence*. If the limit exists it may be calculated by:

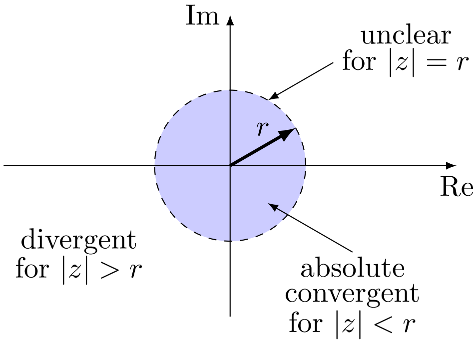
$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

or

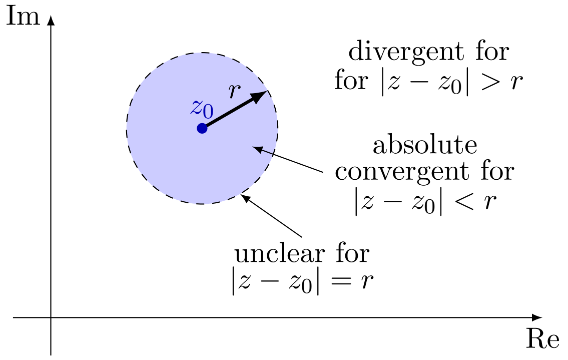
$$r = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}$$

◁

Remark: On the complex plane the set of all z where the power series is convergent represents a circular disk. The power series is convergent for all z within this disk and divergent for all z outside this disk. Along the edge of this disk convergence is unclear.



For a power series with offset z_0 (see definition 7.1) the disk shifts with its centre to z_0:



Example 7.2. We look at the power series in example 7.1:

- $f_1(z)$ is absolute convergent for all $z \in \mathbb{C}$. (A finite polynomial is convergent for all $z \in \mathbb{C}$.)
- $f_2(z)$ is absolute convergent for all $|z| < 1$ (ratio or root test).
- $f_3(z)$ is absolute convergent for $z = 0$ only (ratio or root test).
- $f_4(z)$ is absolute convergent for all $z \in \mathbb{C}$ (ratio or root test).
- $f_5(z)$ is absolute convergent for all $|z| < 1$ (root test).
- $f_6(z)$ is absolute convergent for all $|z| < 2$ (root test).

◁

7.3 Exponential function

Definition 7.4 (Exponential function)**.** With $z \in \mathbb{C}$ we define the *exponential function* with:

$$\begin{aligned}\exp(z) &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \\ &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots\end{aligned}$$

◁

Theorem 7.5 (Calculating with exponential function)**.** With $e \in \mathbb{R}$, $e = \exp(1)$ and $z \in \mathbb{C}$ we have:

$$e^z = \exp(z)$$

With $z_1, z_2 \in \mathbb{C}$ we have:

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$$

◁

Remark: Especially for $z \in \mathbb{C}$, $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $z = a + \mathrm{j}b$ we have

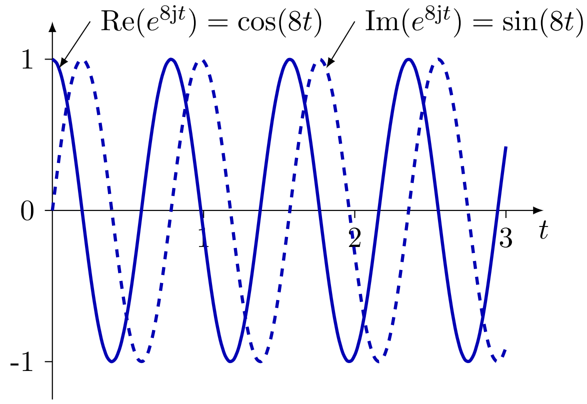
$$e^z = e^a \cdot e^{\mathrm{j}b}$$

where e^a is a damping factor over $-a$ and $e^{\mathrm{j}b}$ is oscillating over b with amplitude one.

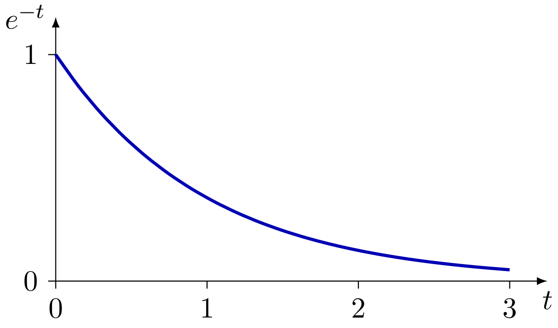
Example 7.3. We want to analyse the function

$$f(t) = e^{(-1+8\mathrm{j})t} = e^{-t} \cdot e^{8\mathrm{j}t}$$

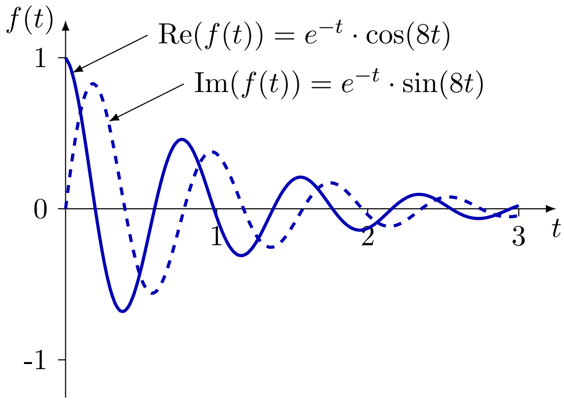
for $t \in \mathbb{R}$, $0 \le t \le 3$. The imaginary part of the exponent results in an osculating value. For $0 \le t \le 3$ the imaginary part of the exponent ranges from 0 to 24 which equals almost four cycles ($\frac{24}{2\pi} \approx 3.82$).



The real part of the exponent is a damping factor from e^0 to e^{-3}:



The function $f(t)$ is the product of the two exponential functions:



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7.4 Trigonometric functions

Definition 7.6 (Sine and cosine function)**.** With $z \in \mathbb{C}$ we define the sine and cosine function with:

$$\begin{aligned}\sin(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \\ &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \dots \\ \cos(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \\ &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \dots\end{aligned}$$

◁

Remark: Due to the odd exponents of z the sine shows point symmetry

$$\sin(z) = -\sin(-z)$$

whereas the cosine with its even powers is axial symmetric:

$$\cos(z) = \cos(-z)$$

Remark: The definitions for the sine and cosine function show similarities to the definition of the exponential function. If we take the exponential function with an imaginary argument we get

$$\begin{aligned}e^{\mathrm{j}z} &= \sum_{k=0}^{\infty} \frac{(\mathrm{j}z)^k}{k!} \\ &= 1 + \mathrm{j}z + \frac{(\mathrm{j}z)^2}{2!} + \frac{(\mathrm{j}z)^3}{3!} + \frac{(\mathrm{j}z)^4}{4!} + \dots \\ &= 1 + \mathrm{j}z - \frac{z^2}{2!} - \mathrm{j}\frac{z^3}{3!} + \frac{z^4}{4!} + \mathrm{j}\frac{z^5}{5!} - \dots \\ &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right) \\ &\quad + \mathrm{j}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right) \\ &= \cos z + \mathrm{j}\sin z\end{aligned}$$

which is *Euler's formula*:

Theorem 7.7 (Euler's formula)**.** For $z \in \mathbb{C}$ we have

$$e^{\mathrm{j}z} = \cos z + \mathrm{j}\sin z$$

◁

Remark: Due to the symmetry of the sine and cosine function we have for a negative exponent:

$$e^{-\mathrm{j}z} = \cos(-z) + \mathrm{j}\sin(-z) = \cos z - \mathrm{j}\sin z$$

This leads us to the weighted sums of exponential functions to express the sine or cosine function:

Theorem 7.8 (Sine and cosine by exponential function)**.** With $z \in \mathbb{C}$ we have

$$\begin{aligned}\sin z &= \frac{e^{\mathrm{j}z} - e^{-\mathrm{j}z}}{2\mathrm{j}} \\ \cos z &= \frac{e^{\mathrm{j}z} + e^{-\mathrm{j}z}}{2}\end{aligned}$$

◁

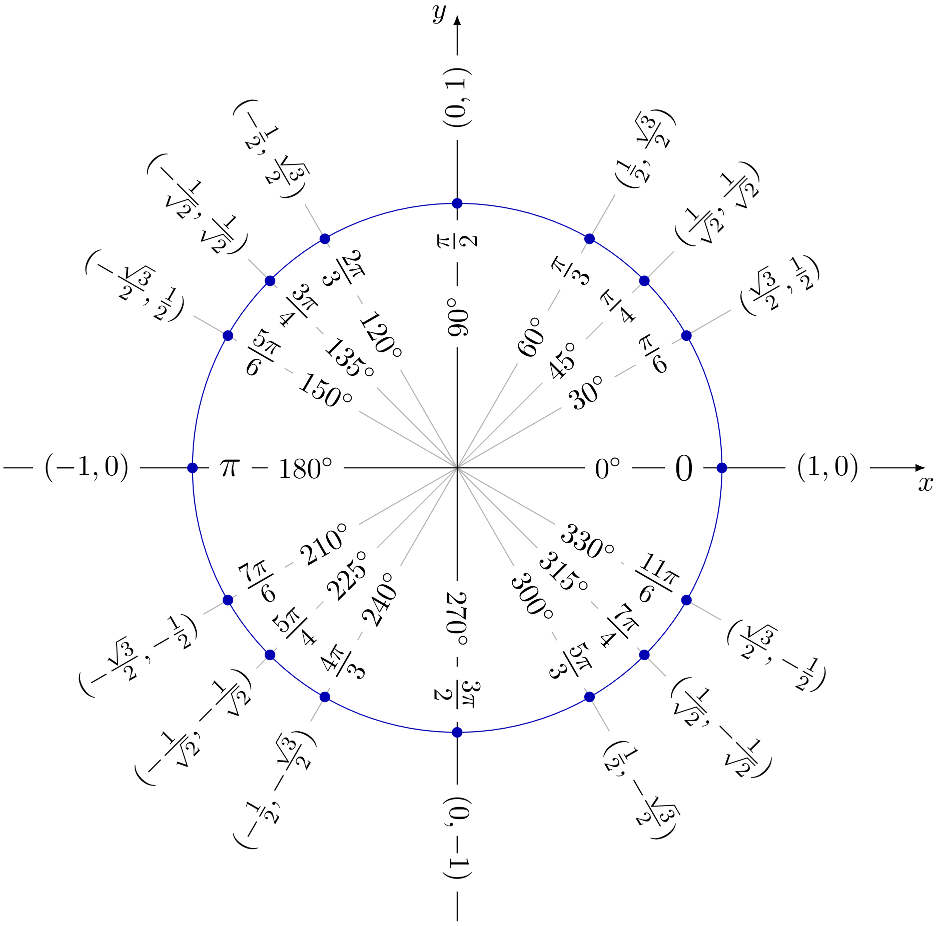


Figure 7.1: Coordinates for some angles on a unit circle, i.e. $(\cos\varphi, \sin\varphi)$, $\varphi \in \mathbb{R}$.

Remark: Now we can derive the addition theorem for sine and cosine, e.g.

$$\begin{aligned}
2\mathrm{j}\sin(a+b) &= e^{\mathrm{j}(a+b)} - e^{-\mathrm{j}(a+b)} \\
&= e^{\mathrm{j}a}e^{\mathrm{j}b} - e^{-\mathrm{j}a}e^{-\mathrm{j}b} \\
&= (\cos a + \mathrm{j}\sin a)(\cos b + \mathrm{j}\sin b) \\
&\quad - (\cos a - \mathrm{j}\sin a)(\cos b - \mathrm{j}\sin b) \\
&= \cos a\cos b + \mathrm{j}\cos a\sin b \\
&\quad + \mathrm{j}\sin a\cos b - \sin a\sin b \\
&\quad - \cos a\cos b + \mathrm{j}\cos a\sin b \\
&\quad + \mathrm{j}\sin a\cos b + \sin a\sin b \\
&= 2\mathrm{j}\cos a\sin b + 2\mathrm{j}\sin a\cos b \\
\sin(a+b) &= \cos a\sin b + \sin a\cos b
\end{aligned}$$

Theorem 7.9 (Addition theorem)**.** With $a, b \in \mathbb{C}$ we have:

$$\begin{aligned}
\sin(a \pm b) &= \sin a \cdot \cos b \pm \cos a \cdot \sin b \\
\cos(a \pm b) &= \cos a \cdot \cos b \mp \sin a \cdot \sin b
\end{aligned}$$

◁

Definition 7.10 (Tangent, cotangent)**.** With $z \in \mathbb{C}$ and $\cos(z) \neq 0$ we define:

$$\tan(z) = \frac{\sin(z)}{\cos(z)}$$

With $z \in \mathbb{C}$ and $\sin(z) \neq 0$ we define:

$$\cot(z) = \frac{\cos(z)}{\sin(z)}$$

◁

Remark: Some easy to memorize values for sine, cosine, tangent and cotangent for real angles:

angle in deg	angle in rad	sin	cos	tan	cot
0	0	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{4}}{2} = 1$	0	undef.
30	$\frac{\pi}{6}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$	$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$	1	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
90	$\frac{\pi}{2}$	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{0}}{2} = 0$	undef.	0

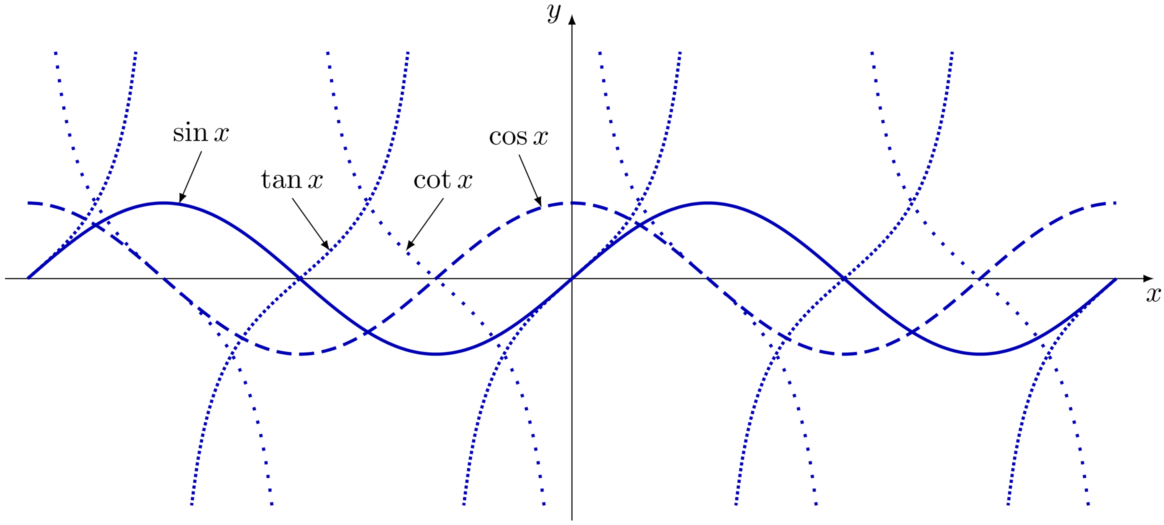


Figure 7.2: Sine, cosine, tangent and cotangent for real arguments in the range $-2\pi \dots 2\pi$.

Due to symmetry and periodicity we can derive the function values all these angles plus $\frac{n\pi}{2}$, $n \in \mathbb{Z}$. E.g. for the sine function we have:

$$\begin{aligned}
\sin\varphi &= -\sin(-\varphi)\\
\sin\varphi &= -\sin(\varphi+\pi)\\
\sin\varphi &= \sin(\varphi+2n\pi)
\end{aligned}$$

Similar relations (but not exactly the same) hold for the cosine, tangent and cotangent functions.

7.5 Problems

Problem 7.1: Which of the following series are power series?

$$\begin{aligned}
f_1(z) &= \sum_{k=0}^{\infty} \frac{(-z)^k}{k}\\
f_2(z) &= \sum_{k=0}^{\infty} (z^k + z^{-k})\\
f_3(z) &= \sum_{k=0}^{\infty} \frac{(z+\sqrt{z})^k}{k}\\
f_4(z) &= \sum_{k=2}^{\infty} \frac{(z+2)^k}{z^2+4z+4}
\end{aligned}$$

Problem 7.2: Find the radius of convergence for the following power series:

$$\begin{aligned}
f_1(z) &= \sum_{k=0}^{\infty} z^k\\
f_2(z) &= \sum_{k=0}^{\infty} \frac{z^k}{3^k}\\
f_3(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k^2} z^k\\
f_4(z) &= \sum_{k=0}^{\infty} (-2z+2\mathrm{j})^k\\
f_5(z) &= \sum_{k=0}^{\infty} \frac{(4z-4)^{k+1}(z+1)}{z^2-1}
\end{aligned}$$

Problem 7.3: Plot the region of convergence on the complex plane for $f_4(z)$ and $f_5(z)$ of the previous problem.

Problem 7.4: Find the radius of convergence for the exponential function.

Problem 7.5: For $t \in \mathbb{R}$ plot the following functions in the range $0 \geq t \geq \pi$:

$$\begin{aligned}
f_1(z) &= e^{t/\pi} & f_2(t) &= e^{-2t/\pi}\\
f_3(t) &= e^{2\mathrm{j}t} & f_4(t) &= e^{-3\mathrm{j}t}\\
f_5(t) &= e^{(1/\pi+10\mathrm{j})t} & f_6(t) &= e^{-(1/\pi+6\mathrm{j})t}
\end{aligned}$$

Problem 7.6: Show that the following equations hold:

$$\begin{aligned}
\sin(a-b) &= \cos a \sin b - \sin a \cos b\\
\cos(a+b) &= \cos a \cos b - \sin a \sin b
\end{aligned}$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

Problem 7.7: Show with Euler's formula that the following holds for any $z \in \mathbb{C}$:

$$\sin^2 z + \cos^2 z = 1$$

Problem 7.8: Express the following terms by exponential functions only and try to reduce them to a minimum:

$$\begin{aligned}
f_1(\varphi) &= \cos\varphi - \mathrm{j}\sin\varphi \\
f_2(\varphi) &= \sin\varphi + \mathrm{j}\cos\varphi \\
f_3(\varphi) &= 2\mathrm{j}\sin\varphi + e^{-\mathrm{j}\varphi} \\
f_4(\varphi) &= \cos\varphi - \tfrac{1}{2}e^{\mathrm{j}\varphi} \\
f_5(\varphi) &= \tfrac{1}{2}\mathrm{Re}(e^{\mathrm{j}\varphi}) + \frac{\mathrm{j}\sin\varphi}{2} \\
f_6(\varphi) &= \cos^2\varphi + \mathrm{Im}(e^{\mathrm{j}\varphi})\sin\varphi
\end{aligned}$$

Problem 7.9: Simplify the following terms (without a pocket calculator):

$$\begin{aligned}
x_1 &= \sin\tfrac{\pi}{4}\cos\tfrac{\pi}{4} & x_2 &= \tan\tfrac{\pi}{4}\cot\tfrac{\pi}{4} \\
x_3 &= \tan 1 \cot 1 & x_4 &= \frac{\cot 2}{\cos 2}\sin 2 \\
x_5 &= \cos\tfrac{\pi}{6}\tan\tfrac{\pi}{3} & x_6 &= \cot\tfrac{\pi}{3}\cos\tfrac{\pi}{6}
\end{aligned}$$

8 Functions

8.1 Definition of functions

In chapter 1 we saw already a broad definition for a function, i.e. there the domain and image where an arbitrary set. In calculus we concentrate mainly on real and complex numbers and, hence, the domain and image are subsets of real or complex numbers.

Definition 8.1 (Function)**.** Let $D \subseteq \mathbb{R}$ or $D \subseteq \mathbb{C}$ and $x \in D$.

- A *real-valued function* maps every element of D to exactly one element of $\mathbb{R}$ and we write:
$$f : \begin{cases} D \to \mathbb{R} \\ x \mapsto f(x) \end{cases}$$
- A *complex-valued function* maps every element of D to exactly one element of $\mathbb{C}$ and we write:
$$f : \begin{cases} D \to \mathbb{C} \\ x \mapsto f(x) \end{cases}$$

◁

Example 8.1.

- $f : \begin{cases} \mathbb{C} \to \mathbb{R} \\ x \mapsto c \end{cases}$ for $c \in \mathbb{R}$
  is a real-valued *constant function*.
- $\mathrm{id}_{\mathbb{C}} : \begin{cases} \mathbb{C} \to \mathbb{C} \\ x \mapsto x \end{cases}$
  is the complex-valued *identity function*.
- $\mathrm{abs} : \begin{cases} \mathbb{C} \to \mathbb{R} \\ x \mapsto |x| = \sqrt{\mathrm{Re}^2(x) + \mathrm{Im}^2(x)} \end{cases}$
  is the real-valued *absolute function*.
- $p : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto p(x) = \sum_{k=0}^{n} a_k x^k \end{cases}$
  is the real-valued polynomial of n^{th} order with coefficients $a_0, a_1, \dots, a_n \in \mathbb{R}$.
- $\exp : \begin{cases} \mathbb{C} \to \mathbb{C} \\ x \mapsto \exp(x) \end{cases}$
  is the complex-valued exponential function.

◁

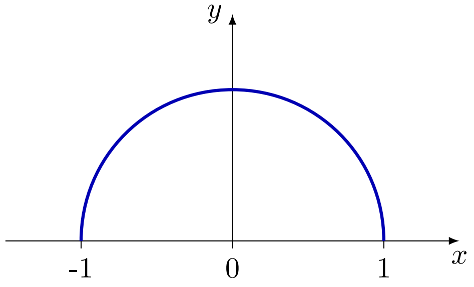
Remark: There are different ways to describe a function:

1. *analytical* either in *implicit* form $f(x, y) = 0$, e.g.
$$x^2 + y^2 - 1 = 0$$
   or in *explicit* form $y = f(x)$, e.g.
$$y = \sqrt{1 - x^2}$$
2. by a table, e.g.

x	y
-1.0	0.000
-0.8	0.600
-0.6	0.800
-0.4	0.917
-0.2	0.980
0.0	1.000
0.2	0.980
0.4	0.917
0.6	0.800
0.8	0.600
1.0	0.000

3. as a set, e.g.
$$\{(x, y) \in [-1, 1] \times \mathbb{R} \mid y = \sqrt{1 - x^2}\}$$
4. with parameter, e.g.
$$x = \cos t, \quad y = \sin t \quad \text{for } t \in [0, \pi]$$

The examples describe the function illustrated in the following graph:



8.2 Properties of functions

Definition 8.2 (Monotonicity)**.** A real-valued function $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$ is called

- *increasing* if $f(a) \geq f(b)$
- *decreasing* if $f(a) \leq f(b)$
- *strictly increasing* if $f(a) > f(b)$
- *strictly decreasing* if $f(a) < f(b)$

for all $a, b \in D$ with $a > b$. If a function is (strictly) increasing or (strictly) decreasing we call it (strictly) *monotone*. ◁

Definition 8.3 (Bounded function)**.** Let $D \in \mathbb{R}$ or $D \in \mathbb{C}$ and f be a real- or complex-valued function over D. We say f is bounded if there exist an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in D$:

$$\exists M \in \mathbb{R}\, \forall x \in D : |f(x)| \leq M$$

◁

Definition 8.4 (Symmetry)**.** Let $D \subseteq \mathbb{R}$ or $D \subseteq \mathbb{C}$ be the domain of a real- or complex-valued function f. The function f shows

- *reflection symmetry* if $f(-x) = f(x)$
- *point symmetry* if $f(-x) = -f(x)$

◁

Definition 8.5 (Periodicity)**.** Let $D \subseteq \mathbb{R}$ or $D \subseteq \mathbb{C}$ be the domain of a real- or complex-valued function f. The function f is said to be periodic if there exist a real or complex non-zero constant T with

$$f(x+T) = f(x) \qquad \text{for all } x+T, x \in D$$

◁

Example 8.2.

- $\exp : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \exp(x) \end{cases}$

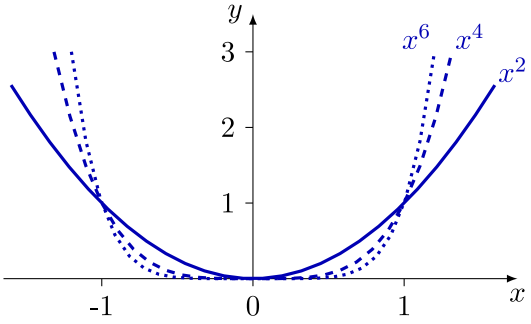
  is strictly increasing, is not bounded, is not symmetric and is not periodic.

- $\sin : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \sin(x) \end{cases}$

  is not monotone, is bounded, is point symmetric and is periodic over 2π.

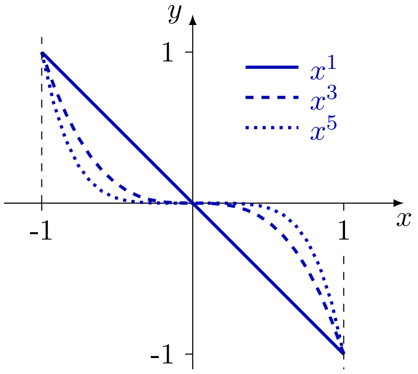
- $p : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^{2n}, n \in \mathbb{N} \end{cases}$

  is not monotone, is not bounded, shows reflection symmetry and is not periodic.



- $p : \begin{cases} [-1,1] \to \mathbb{R} \\ x \mapsto -x^{2n-1}, n \in \mathbb{N} \end{cases}$

  is strictly decreasing, is bounded, shows point symmetry and is not periodic.



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Definition 8.6 (Zero of a function)**.** Let $D \subseteq \mathbb{R}$ or $D \subseteq \mathbb{C}$ be the domain of a real- or complex-valued function f and $x' \in D$. For

$$f(x') = 0$$

we say x' is a *zero* of f. ◁

Definition 8.7 (Extrema)**.** Let $D \subseteq \mathbb{R}$ or $D \subseteq \mathbb{C}$ be the domain of a real-valued function f with $x' \in D$ and the epsilon neighbourhood $U_\varepsilon(x')$. We say x' is a

- *local!maximum* if

  $$f(x') \geq f(x) \qquad \text{for all } x \in U_\varepsilon(x') \cap D$$

- *local!minimum* if

  $$f(x') \leq f(x) \qquad \text{for all } x \in U_\varepsilon(x') \cap D$$

- *global!maximum* if

$$f(x') \geq f(x) \qquad \text{for all } x \in D$$

- *global!minimum* if

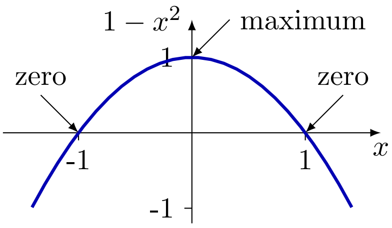
$$f(x') \leq f(x) \qquad \text{for all } x \in D$$

If x' is a global/local maximum or minimum we call it global/local *extremum*. ◁

Example 8.3.

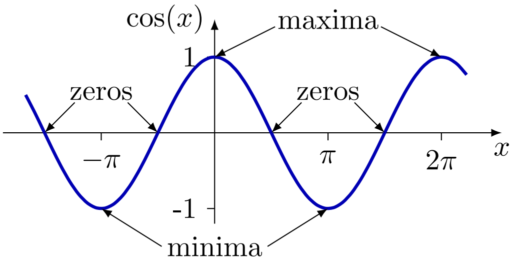
- $p : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto 1 - x^2 \end{cases}$

  has zeros at $x = \pm 1$ and a maximum at $x = 0$.



- $\cos : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \cos(x) \end{cases}$

  has zeros at $x = n\pi + \frac{\pi}{2}$, maxima at $x = 2n\pi$ and minima at $x = 2n\pi + \pi$ for $n \in \mathbb{Z}$.



- $p : \begin{cases} \mathbb{C} \to \mathbb{C} \\ x \mapsto x^2 + 1 \end{cases}$

  has zeros at $x = \pm$j. (Extrema are not defined for complex-valued functions.)

- $\ln : \begin{cases} \mathbb{R}_{>0} \to \mathbb{R} \\ x \mapsto \ln(x) \end{cases}$

  has a zero at $x = 1$ and no extremum.

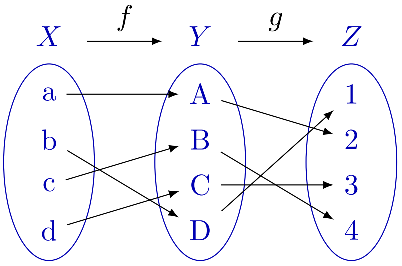
◁

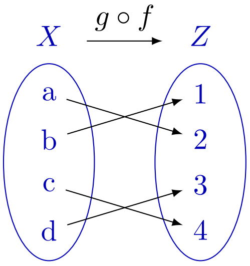
8.3 Composed and inverse functions

Definition 8.8 (Function composition)**.** Let X, Y and Z be sets and $f : X \to Y$ and $g : Y \to Z$ be two functions. We define

$$g \circ f : \begin{cases} X \to Z \\ x \mapsto g(f(x)) \end{cases}$$

as the *composition* of f and g. ◁





Remark: For the composition commutativity does not hold, i.e. $g \circ f \neq f \circ g$. The functions must be applied from right to left, i.e. for $(g \circ f)(x)$ the function $f(x)$ is applied first and its value serves as input for g: $g(f(x))$.

Example 8.4.

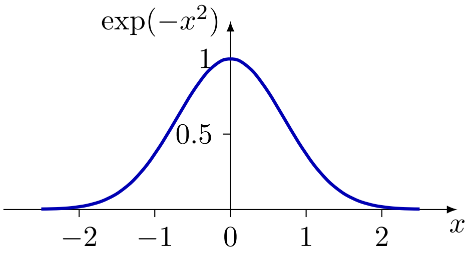
$$g \circ f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \exp(-x^2) \end{cases}$$

is the composition of

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto -x^2 \end{cases}$$

and

$$g : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \exp(x) \end{cases}$$



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Definition 8.9 (Inverse function)**.** Let X and Y be sets and $f : X \to Y$ be a bijective function. We define

$$f^{-1} : \begin{cases} Y \to X \\ f(x) \mapsto x \end{cases}$$

as the *inverse function*. ◁

Remark: The inverse is not the reciprocal:

$$f^{-1} \neq \frac{1}{f(x)}$$

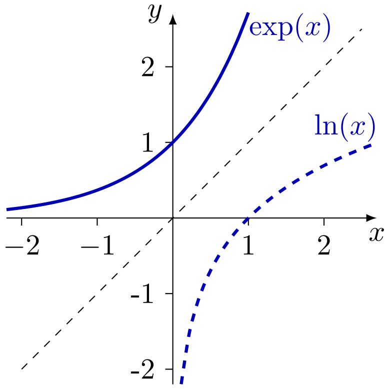
In a Cartesian diagram the inverse is the mirror image at the diagonal line $x = y$.

Example 8.5.

- $\exp : \begin{cases} \mathbb{R} \to \mathbb{R}_{>0} \\ x \mapsto \exp(x) \end{cases}$

  has the inverse:

  $\ln : \begin{cases} \mathbb{R}_{>0} \to \mathbb{R} \\ x \mapsto \ln(x) \end{cases}$



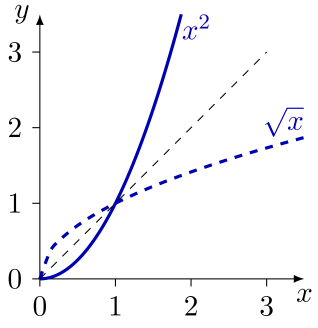
- $p : \begin{cases} \mathbb{R} \to \mathbb{R}_{\geq 0} \\ x \mapsto x^2 \end{cases}$

  has no inverse since it is not bijective.

- $p : \begin{cases} \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \\ x \mapsto x^2 \end{cases}$

  has the inverse:

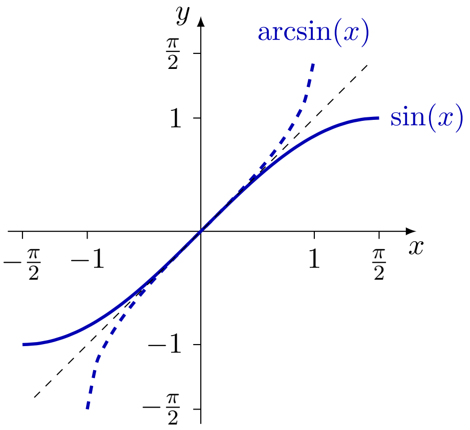
  $\text{sqrt} : \begin{cases} \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \\ x \mapsto \sqrt{x} \end{cases}$



- $\sin : \begin{cases} [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1] \\ x \mapsto \sin(x) \end{cases}$

  has the inverse:

  $\arcsin : \begin{cases} [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}] \\ x \mapsto \arcsin(x) \end{cases}$



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8.4 Continuity

Definition 8.10 (Limit of a function)**.** Let $D \subseteq \mathbb{R}$ be the domain of a real- or complex-valued function f. With $x_0 \in D$, $x \in D \setminus \{x_0\}$ and $y_0 \in \mathbb{R}$ or $y_0 \in \mathbb{C}$ the expression

$$\lim_{x \to x_0} f(x) = y_0$$

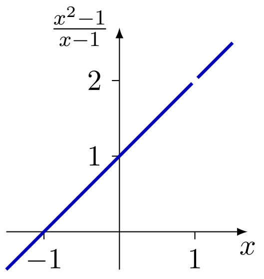
means that $f(x)$ can be as close as desired to y_0 by making x sufficient close to x_0. ◁

Example 8.6. The function

$$f : \begin{cases} \mathbb{R} \setminus \{1\} \to \mathbb{R} \\ x \mapsto \frac{x^2-1}{x-1} \end{cases}$$

is not defined for $x = 1$. However the limit is:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$



◁

Definition 8.11 (One-sided limit)**.** Let $D \subseteq \mathbb{R}$ be the domain of a real- or complex-valued function f, $x_0 \in D$ and $x \in D \setminus \{x_0\}$. We define the

- *left-sided limit* as the limit of $f(x)$ with $x < x_0$:
$$\lim_{x \to x_0^-} f(x)$$
- *right-sided limit* as the limit of $f(x)$ with $x > x_0$:
$$\lim_{x \to x_0^+} f(x)$$

◁

Example 8.7. The *sign function*

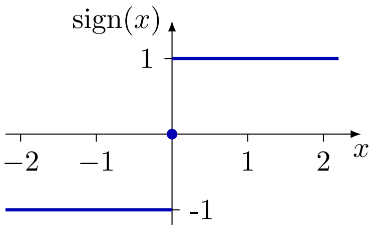
$$\text{sign} : \begin{cases} \mathbb{R} \to \mathbb{R} & \\ x \mapsto 1 & \text{for } x > 0 \\ x \mapsto 0 & \text{for } x = 0 \\ x \mapsto -1 & \text{for } x < 0 \end{cases}$$

has the right-side limit

$$\lim_{x \to 0^+} \text{sign}(x) = 1$$

and the left-side limit

$$\lim_{x \to 0^-} \text{sign}(x) = -1$$



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Definition 8.12 (Continuity)**.** Let $D \subseteq \mathbb{R}$ be the domain of a real- or complex-valued function f, $x_0 \in D$ and $x \in D \setminus \{x_0\}$. We say f is *continuous* in x_0 if

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$$

If f is continuous in all $x \in D$ we say f is a *continuous function.* ◁

Remark: Visually speaking, if you are able to draw a function without lifting the pen it is continuous.

Example 8.8.

- $\exp(x)$, $\sin(x)$ and $\cos(x)$ are continuous functions in $\mathbb{R}$.
- $\tan(x)$ is continuous in $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- $\text{sign}(x)$ (see previous example) is not continuous in $x = 0$.
- All polynomials are continuous.

◁

Theorem 8.13 (Intermediate value theorem)**.** Let f be a real-valued continuous function over $[a, b]$, $a < b$ and $f(a) < f(b)$. For any value $y_0 \in [f(a), f(b)]$ there exists an $x_0 \in [a, b]$ with $f(x_0) = y_0$.

$$\forall y_0 \in [f(a), f(b)]\ \exists x_0 \in [a, b] : f(x_0) = y_0$$

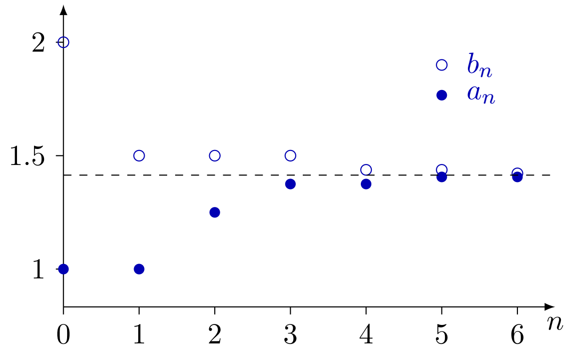
The corresponding holds for $f(a) > f(b)$. ◁

Remark: Especially for $f(a) \cdot f(b) < 0$ (i.e. one factor is positive and the other negative) the intermediate value theorem states that there must be a at least one zero between a and b.

Example 8.9. The *bisection method* is an algorithm to find zeros of an arbitrary function $f : \mathbb{R} \to \mathbb{R}$. First two values a_0 and b_0 are required where $f(a_0) \cdot f(b_0) < 0$, say, $f(a_0) < 0$ and $f(b_0) > 0$. Then the function value at $c_0 = \frac{a_0 + b_0}{2}$ is determined. If $f(c_0) < 0$ then $a_1 = c_0$ and $b_1 = b_0$, else $a_1 = a_0$ and $b_1 = c_0$. With a_1 and b_1 we repeat the same process and approach the zero of f.

E.g. with the function $f(x) = x^2 - 2$ and $a = 1$ and $b = 2$ we get the following table:

n	a_n	b_n	c_n
0	1	2	1.5
1	1	1.5	1.25
2	1.25	1.5	1.375
3	1.375	1.5	1.438
4	1.375	1.438	1.406
5	1.406	1.438	1.422
6	1.406	1.422	1.414
		…	



The bisection method is simple and robust, but rather slow. In computer science it is used to derive an approximation as an input for a more efficient algorithm, e.g. Newton's method. ◁

8.5 Problems

Problem 8.1: Plot the following functions:

1. $f : \begin{cases} [0, \pi] \to \mathbb{R} \\ x \mapsto \sin(x) \end{cases}$
2.

x	0	0.2	0.4	0.6	0.8	1
$f(x)$	0	0.04	0.16	0.36	0.64	1

3. $\{(x, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \mid y = \sqrt{x}\}$
4. $x = \pm\sqrt{t}$, $y = \exp(-t)$ for $t \in [0, 4]$.

Problem 8.2: Give statements on monotonicity, boundary, symmetry, periodicity, zeros and extrema for the following functions:

1. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto e^{-x} \end{cases}$
2. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto e^{-x^2} \end{cases}$
3. $f : \begin{cases} \mathbb{R} \to \mathbb{C} \\ x \mapsto e^{\mathrm{j}x} \end{cases}$
4. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \mathrm{Im}(e^{\mathrm{j}x}) \end{cases}$
5. $f : \begin{cases} \mathbb{R}_{<0} \to \mathbb{R} \\ x \mapsto \ln(-x) \end{cases}$
6. $f : \begin{cases} (0, \pi) \to \mathbb{R} \\ x \mapsto \cot(x) \end{cases}$
7. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^5 + x^3 + x \end{cases}$

Problem 8.3: Find the inverse of the following functions:

1. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 - 1 \end{cases}$
2. $f : \begin{cases} \mathbb{R}_{\leq 0} \to \mathbb{R}_{\geq 2} \\ x \mapsto x^2 + 2 \end{cases}$
3. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \cos(x) \end{cases}$
4. $f : \begin{cases} (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R} \\ x \mapsto \tan(x) \end{cases}$

Problem 8.4: Which of the following functions are continuous in their whole domain?

1. $f : \begin{cases} (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R} \\ x \mapsto \tan(x) \end{cases}$
2. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto |x| \end{cases}$
3. The sign-function as in example 8.7
4. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto (x - 1)(x + 1) \end{cases}$

Problem 8.5: A continuous function in $\mathbb{R}$ has the values $f(1) = -1$ and $f(2) = 1$. Give a statement on zeros of f.

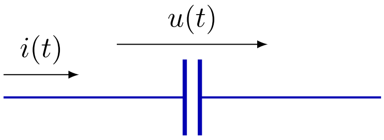
9 Differential calculus

9.1 Introduction

In engineering we are not only focussed on values at a given time or position but also in their dynamic behaviour. E.g. for speed control the police is not so much interested in the position of a car *at* a given time but rather in the speed of the car, i.e. the *change* of position *per* time.

Another example is a capacitor in electrical engineering: We investigate the change of voltage to determine the electrical current that caused the voltage change.

Let's assume a capacitor with capacity C measured in Farad, F. The capacitor is charged by a current i which is a function over time t and we write in short $i(t)$. The charging current results in a voltage over the capacitor u which also varies over time t and we write $u(t)$.



By measuring the voltage u we can evaluate the current i. If the voltage increases with constant speed over, say, two seconds by 4 V on a capacitor with 0.01 F, then the charging current is

$$i = C\frac{\Delta u}{\Delta t} = 0.01\,\mathrm{F}\cdot\frac{4\,\mathrm{V}}{2\,\mathrm{s}} = 0.02\,\mathrm{A} = 20\,\mathrm{mA}$$

If the current varies over time we need to find out the current i at a ginven time t. To do so we reduce the time interval towards zero

$$i(t) = \lim_{\Delta t\to 0} C\frac{u(t+\Delta t)-u(t)}{\Delta t} = C\frac{\mathrm{d}u}{\mathrm{d}t}(t)$$

Let's assume the function for the voltage is a sine-wave with frequency f and amplitude $\hat{u}$ i.e.

$$u(t) = \hat{u}\sin(2\pi f t)$$

For the current $i(t)$ we get:

$$\begin{aligned} i(t) &= C\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}\sin(2\pi f t) \\ &= \hat{u}2\pi f C\cos(2\pi f t) \\ &= \hat{\imath}\cos(2\pi f t) \end{aligned}$$

with $\hat{\imath} = \hat{u}2\pi f C$.

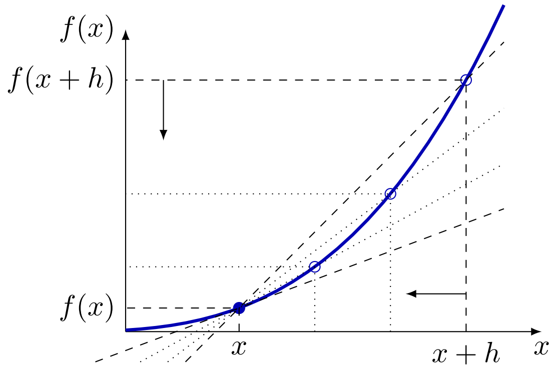
By the end of this chapter you will be able to apply the required mathematics to solve this kind of equations.

9.2 Differentiability

Definition 9.1 (Differentiability)**.** Let $D \subseteq \mathbb{R}$. We say $f : D \to \mathbb{R}$ is *differentiable* at $x \in D$ if the limit

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

exist. We call the limit the *derivative* of f at x. If f is differentiable at all $x \in D$ we say f is differentiable. ◁



Remark: With $x' = x+h$ we may express the derivative as

$$\lim_{x'\to x}\frac{f(x')-f(x)}{x'-x}$$

For the derivative of f we write $\frac{\mathrm{d}f}{\mathrm{d}x}(x)$, $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$ or simply $f'(x)$. Applications in physics often deal with functions over time t and physicists write in short $\frac{\mathrm{d}f}{\mathrm{d}t}(t) = \dot{f}(t) = \dot{f}$ where the dot indicates the derivative w.r.t. time.

The x in brackets and denominator of $\frac{\mathrm{d}f}{\mathrm{d}x}(x)$ seem redundant. However, later we will have situations to differentiate functions to x which have no dependency to x but, say, y. There we write $\frac{\mathrm{d}f}{\mathrm{d}x}(y)$.

If the context is clear we may skip the x in brackets and write $\frac{\mathrm{d}f}{\mathrm{d}x}$.

Example 9.1.

- $sq : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ is differentiable.
- $\sin : \mathbb{R} \to \mathbb{R}, x \mapsto \sin(x)$ is differentiable.
- $abs : \mathbb{R} \to \mathbb{R}, x \mapsto |x|$ is not differentiable at 0.

◁

Theorem 9.2 (Differentiability implies continuity)**.** Every differentiable function is continuous. ◁

Remark: Since it is an implication it does not necessary mean that every continuous function is differentiable, see the following example.

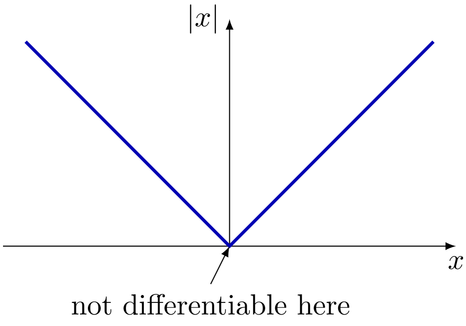
Example 9.2. Is the function of the absolute differentiable at zero?

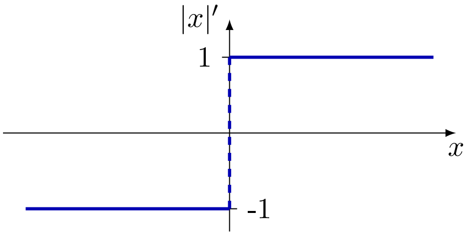
$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto |x| \end{cases}$$

The function is continuous in $\mathbb{R}$. We take the left- and right-sided limit to test differentiability at zero:

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|-|0|}{h} = -1$$
$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|-|0|}{h} = +1$$

We get two different values, hence, although being continuous the function of the absolute is not differentiable at zero.





◁

9.3 Some derivatives

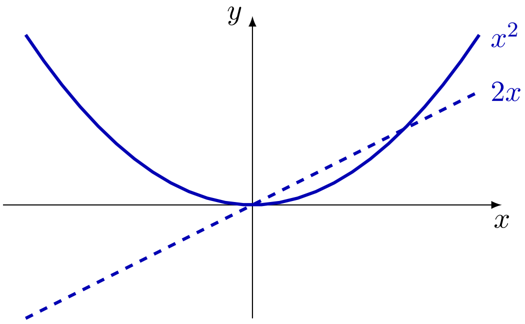
In this section we want to evaluate some typical derivatives.

Example 9.3. What is the derivative of

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{cases} \quad ?$$

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \underbrace{\lim_{h\to 0} 2x}_{=2x} + \underbrace{\lim_{h\to 0} h}_{=0} = 2x \end{aligned}$$

Hence, the derivative of x^2, $x \in \mathbb{R}$ is $2x$.



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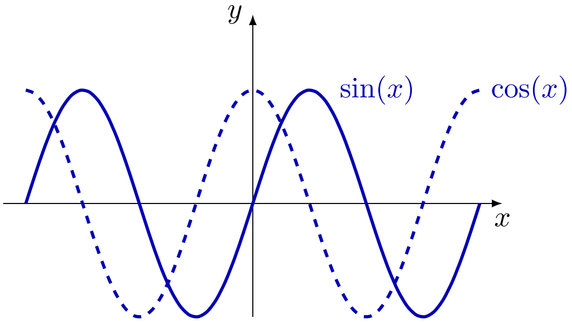
Example 9.4. What is the derivative of

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \sin(x) \end{cases} \quad ?$$

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)\overbrace{\cos(h)}^{\to 1} + \cos(x)\overbrace{\sin(h)}^{\to h} - \sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x) + h\cos(x) - \sin(x)}{h} \\ &= \lim_{h\to 0} \cos(x) = \cos(x) \end{aligned}$$

Hence, the derivative of $\sin(x)$, $x \in \mathbb{R}$ is $\cos(x)$.



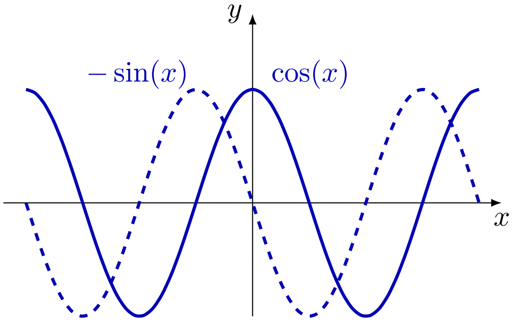
◁

Example 9.5. What is the derivative of

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \cos(x) \end{cases} \qquad ?$$

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\overbrace{\cos(h)}^{\to 1} - \sin(x)\overbrace{\sin(h)}^{\to h} - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x) - h\sin(x) - \cos(x)}{h} \\
&= \lim_{h\to 0} -\sin(x) = -\sin(x)
\end{aligned}$$

Hence, the derivative of $\cos(x)$, $x \in \mathbb{R}$ is $-\sin(x)$.



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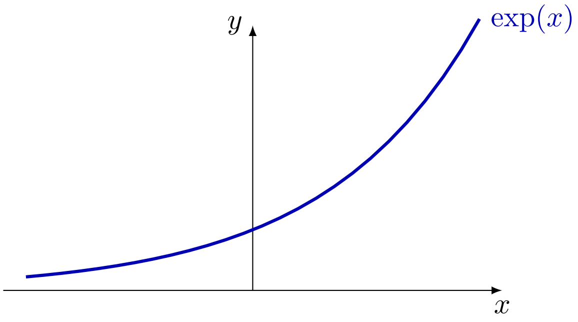
Example 9.6. What is the derivative of

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto e^x \end{cases} \qquad ?$$

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} \\
&= \lim_{h\to 0} \frac{e^x(-1 + 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots)}{h} \\
&= \lim_{h\to 0} e^x \overbrace{\left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \ldots\right)}^{\to 1} = e^x
\end{aligned}$$

Hence, the derivative of e^x, $x \in \mathbb{R}$ is e^x.



◁

Some more of the *basic derivatives* may be found in the upper part of table 9.1.

$f(x)$	$f'(x)$
c	$0 \quad \text{for } c \in \mathbb{R}$
x^c	$cx^{c-1} \quad \text{for } c \in \mathbb{R}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
e^x	e^x
c^x	$c^x \ln c \quad \text{for } c \in \mathbb{R}, c > 0$
$\ln \lvert x \rvert$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x) = \frac{\sin(x)}{\cos(x)}$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$c \cdot f(x)$	$c \cdot f'(x) \quad \text{for } c \in \mathbb{R}$
$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
$f(x) \cdot g(x)$	$f'(x) \cdot g(x) + f(x) \cdot g'(x)$
$\frac{f(x)}{g(x)}$	$\frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$
$f(g(x))$	$f'(g) \cdot g'(x)$

Table 9.1: Some important derivatives.

9.4 Calculation rules for derivatives

Theorem 9.3 (Basic operations on derivatives)**.** Let f and g be differentiable functions and $c \in \mathbb{R}$. Then we have

$$\begin{aligned}
(f(x) \pm g(x))' &= f'(x) \pm g'(x) \\
(c \cdot f(x))' &= c \cdot f'(x) \\
(f(x) \cdot g(x))' &= f'(x) \cdot g(x) + f(x) \cdot g'(x)
\end{aligned}$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}$$

◁

Example 9.7.

- Let $f(x) = \sin(x) + x^2$, then

$$f'(x) = \tfrac{\mathrm{d}}{\mathrm{d}x}\sin(x) + \tfrac{\mathrm{d}}{\mathrm{d}x}x^2 = \cos(x) + 2x$$

- Let $f(x) = 3x^3$, then

$$f'(x) = 3\tfrac{\mathrm{d}}{\mathrm{d}x}x^3 = 3\cdot 3x^2 = 9x^2$$

- Let $f(x) = \sin(x)\cdot\cos(x)$, then

$$\begin{aligned} f'(x) &= \tfrac{\mathrm{d}\sin(x)}{\mathrm{d}x}\cos(x) + \sin(x)\tfrac{\mathrm{d}\cos(x)}{\mathrm{d}x} \\ &= \cos(x)\cos(x) + \sin(x)(-\sin(x)) \\ &= \cos^2(x) - \sin^2(x) \end{aligned}$$

- Let $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$, then

$$\begin{aligned} f'(x) &= \frac{\tfrac{\mathrm{d}\sin(x)}{\mathrm{d}x}\cos(x) - \sin(x)\tfrac{\mathrm{d}\cos(x)}{\mathrm{d}x}}{\cos^2(x)} \\ &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} \end{aligned}$$

◁

Theorem 9.4 (Chain rule)**.** Let $X, Y, Z \subseteq \mathbb{R}$ and $f : Y \to Z$ and $g : X \to Y$ be two differentiable functions. With $f \circ g = f(g(x))$ as the composition of the two functions we have:

$$(f \circ g)' = (f' \circ g)\cdot g'$$

i.e.

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{\mathrm{d}f}{\mathrm{d}g}(g)\cdot\frac{\mathrm{d}g}{\mathrm{d}x}(x)$$

◁

Remark: Mnemonic: “Outer derivative times inner derivative”

Example 9.8.

- Let $f(x) = \sin(2\pi f x)$, then

$$\begin{aligned} f(g) &= \sin(g) \qquad g(x) = 2\pi f x \\ f'(x) &= f'(g)\cdot g'(x) = \cos(g)\cdot 2\pi f \\ &= 2\pi f\cos(2\pi f x) \end{aligned}$$

- Let $f(x) = \sin^2(x)$, then

$$\begin{aligned} f(g) &= g^2 \qquad g(x) = \sin(x) \\ f'(x) &= f'(g)\cdot g'(x) = 2g\cdot\cos(x) \\ &= 2\sin(x)\cos(x) = \sin(2x) \end{aligned}$$

- Let $f(x) = \sqrt{\ln|2x|}$, then

$$\begin{aligned} f(g) &= \sqrt{g} \qquad g(h) = \ln|h| \qquad h(x) = 2x \\ f'(x) &= f'(g)\cdot g'(h)\cdot h'(x) \\ &= \frac{1}{2\sqrt{g}}\cdot\frac{1}{h}\cdot 2 = \frac{1}{2\sqrt{\ln|2x|}}\cdot\frac{1}{2x}\cdot 2 \\ &= \frac{1}{2x\sqrt{\ln|2x|}} \end{aligned}$$

- Let $f(x) = e^{-\frac{x^2}{2}}$, then

$$f'(x) = e^{-\frac{x^2}{2}}\cdot\frac{-1}{2}\cdot 2x = -x\,e^{-\frac{x^2}{2}}$$

◁

Theorem 9.5 (Mean value theorem)**.** Let $I \subseteq \mathbb{R}$ an interval, $f : I \to \mathbb{R}$ be a differentiable function and $a, b \in I$ with $a < b$. There exist at least one $x \in I$, $a < x < b$ such that

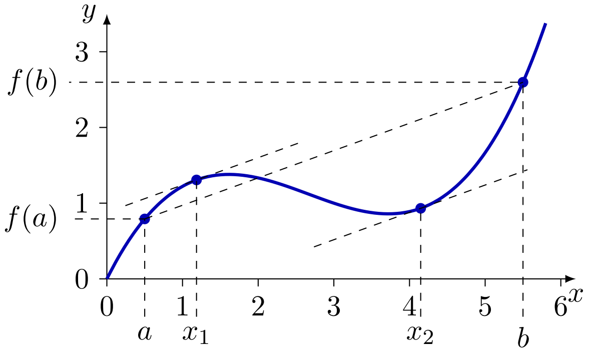
$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

◁

Example 9.9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $x \mapsto f(x) = \frac{1}{9}x^3 - \frac{8}{9}x^2 + 2x$. Let further $a = \frac{1}{2}$ and $b = 5\frac{1}{2}$ be two elements of the domain of f. Then we have:

$$\begin{aligned} f'(x) &= \frac{f(b) - f(a)}{b - a} \\ \frac{1}{3}x^2 - \frac{16}{9}x + 2 &= \frac{2\frac{43}{72} - \frac{19}{24}}{5\frac{1}{2} - \frac{1}{2}} = \frac{13}{36} \\ x^2 - \frac{16}{3}x &= \frac{13}{12} - 6 \\ \left(x - \frac{8}{3}\right)^2 &= \frac{13}{12} - 6 + \frac{64}{9} \\ \left(x - \frac{8}{3}\right)^2 &= \frac{39}{36} - \frac{216}{36} + \frac{256}{36} = \frac{79}{36} \\ x_{1,2} &= \tfrac{8}{3} \pm \tfrac{1}{6}\sqrt{79} \end{aligned}$$

I.e. at position x_1 and x_2 the first derivative of f equals the mean derivative between a and b.



◁

Remark: If a derivative is differentiable it may be differentiated again. We then say *second derivative* and write $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x)$, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x)$ or $f''(x)$.

If a function f is differentiated n-times we then say n^{th} derivative and we write $\frac{\mathrm{d}^n f}{\mathrm{d}x^n}(x)$, $\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x)$ or $f^{(n)}(x)$.

Example 9.10.

- Let $f(x) = x^3 - 2x^2 + 4x - 2$, then

$$\begin{aligned} f'(x) &= 3x^2 - 4x + 4 \\ f''(x) &= 6x - 4 \\ f'''(x) &= 6 \\ f^{(4)}(x) &= 0 \end{aligned}$$

- Let $u(t) = \hat{u}\sin(\omega t)$, then

$$\begin{aligned} u'(t) &= \quad \omega \;\; \hat{u}\cos(\omega t) \\ u''(t) &= -\,\omega^2\,\hat{u}\sin(\omega t) \\ u'''(t) &= -\,\omega^3\,\hat{u}\cos(\omega t) \\ u^{(4)}(t) &= \quad \omega^4\,\hat{u}\sin(\omega t) \end{aligned}$$

- Let $f(t) = e^{k\,t}$, then

$$\begin{aligned} f'(t) &= k \;\; e^{k\,t} \\ f''(t) &= k^2\, e^{k\,t} \\ f'''(t) &= k^3\, e^{k\,t} \\ &\dots \\ f^{(n)}(t) &= k^n\, e^{k\,t} \end{aligned}$$

- Let $f(x) = \ln|x|$, then

$$\begin{aligned} f'(x) &= \frac{1}{x} = x^{-1} \\ f''(x) &= -x^{-2} = \frac{-1}{x^2} \\ f'''(x) &= 2x^{-3} = \frac{2}{x^3} \end{aligned}$$

◁

9.5 Problems

Problem 9.1: Plot the following function:

$$f: \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto |x - n| \text{ for } n - \frac{1}{2} \leq x < n + \frac{1}{2}, n \in \mathbb{Z} \end{cases}$$

- Is this function continuous?
- Is this function differentiable?

Problem 9.2: Evaluate the following derivatives.

1. $\frac{\mathrm{d}}{\mathrm{d}x}(-2x^2)$
2. $\frac{\mathrm{d}}{\mathrm{d}u}3e^u$
3. $\frac{\mathrm{d}}{\mathrm{d}x}(x^5 + \sqrt{2}x^3)$
4. $\frac{\mathrm{d}}{\mathrm{d}x}(x^3 - x^2 + 1)$
5. $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x + \cos y)$
6. $\frac{\mathrm{d}}{\mathrm{d}y}(\sin x + \cos y)$

Problem 9.3: Evaluate the following derivatives.

1. $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + 1)$
2. $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{6}x^9 + \frac{2}{9}x^3 + \frac{1}{5}x^4)$
3. $\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + 2xy + y^2)$
4. $\frac{\mathrm{d}}{\mathrm{d}y}(x^2 - 2xy + y^2)$
5. $\frac{\mathrm{d}}{\mathrm{d}z}(x^2 - y^2)$
6. $\frac{\mathrm{d}}{\mathrm{d}x}x^1y^2z^3$
7. $\frac{\mathrm{d}}{\mathrm{d}y}x^1y^2z^3$
8. $\frac{\mathrm{d}}{\mathrm{d}z}x^1y^2z^3$

Problem 9.4: Evaluate the following derivatives.

1. $\frac{\mathrm{d}}{\mathrm{d}x}(x^{-2} + x^{-3})$
2. $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3})$
3. $\frac{\mathrm{d}}{\mathrm{d}x}(3x^{\frac{2}{3}} - 6x^{-\frac{1}{3}})$
4. $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{2}x^2 - \frac{1}{3}x^{-3})$
5. $\frac{\mathrm{d}}{\mathrm{d}x}(x^{\frac{1}{2}} - x^{-\frac{1}{2}})$
6. $\frac{\mathrm{d}}{\mathrm{d}x}(x^2 - 1)(x^{-2} + 1)$

Problem 9.5: Evaluate the following derivatives.

1. $\frac{\mathrm{d}}{\mathrm{d}x} e^{\mathrm{j}x} \sin x$
2. $\frac{\mathrm{d}}{\mathrm{d}x} \tan x \cos x$
3. $\frac{\mathrm{d}}{\mathrm{d}x} 2x^{\frac{1}{2}} \sin x$
4. $\frac{\mathrm{d}}{\mathrm{d}x} \frac{x^3-x^2+1}{x^2+1}$
5. $\frac{\mathrm{d}}{\mathrm{d}x} \dfrac{x^2}{e^x}$
6. $\frac{\mathrm{d}}{\mathrm{d}x} \dfrac{\sin x}{2^x}$

Problem 9.6: Evaluate the following derivatives.

1. $\frac{\mathrm{d}}{\mathrm{d}x} e^{x^2}$
2. $\frac{\mathrm{d}}{\mathrm{d}t} \hat{u} \sin(2\pi f t)$
3. $\frac{\mathrm{d}}{\mathrm{d}t} \hat{i} e^{\mathrm{j}(2\pi f t+\varphi)}$
4. $\frac{\mathrm{d}}{\mathrm{d}x} \sin^2(x)$
5. $\frac{\mathrm{d}}{\mathrm{d}x} \sqrt{\ln |x^2|}$
6. $\frac{\mathrm{d}}{\mathrm{d}x} \sqrt{\sin(3^x)}$

Problem 9.7: Evaluate the following derivatives ($n \in \mathbb{N}$).

1. $\frac{\mathrm{d}^2}{\mathrm{d}x^2} x^4$
2. $\frac{\mathrm{d}^3}{\mathrm{d}x^3} 3^x$
3. $\frac{\mathrm{d}^n}{\mathrm{d}x^n} \pi^x$
4. $\frac{\mathrm{d}^2}{\mathrm{d}t^2} \hat{u} \sin(2\pi f t)$
5. $\frac{\mathrm{d}^4}{\mathrm{d}x^4} \sin(2x)$
6. $\frac{\mathrm{d}^n}{\mathrm{d}t^n} \hat{u} e^{\mathrm{j}(2\pi f t+\varphi)}$

10 Polynomials and rational functions

10.1 Polynomial function

Definition 10.1 (Polynomial function)**.** Let $n \in \mathbb{N} \cup \{0\}$, $a_k \in \mathbb{C}$, $k = 0, 1, \ldots, n$, $a_n \neq 0$ and $x \in \mathbb{C}$. We then call

$$\begin{aligned} p &= \sum_{k=0}^{n} a_k x^k \\ &= a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \end{aligned}$$

a *polynomial function* or just a *polynomial*. We say p is of n^{th} order and call a_k the *coefficients* of p. $\lhd$

Remark: For our purposes we limit ourself to real coefficients.

Example 10.1.

- $p_1(x) = x^2 - x + 1$ *is* a 2nd order polynomial.
- $p_2(x) = x^5 - 1$ *is* a 5th order polynomial.
- $p_3(x) = x^2 + 3/x + 1$ *is not* a polynomial due to the negative exponent of the second summand.
- $p_4(x) = x^3 - 2x^{1/2} + 3$ *is not* a polynomial since x of the second summand has a non-integer exponent.
- $p_5(x) = 2^{1/2}x^2 + x - 3$ *is* a second order polynomial ($2^{1/2} = \sqrt{2} \in \mathbb{R}$ is a coefficient for x^2).
- $p_6(x) = 1$ *is* a zero order polynomial.

$\lhd$

Theorem 10.2 (Properties of polynomials)**.**

- Polynomials are *continuous.*
- Polynomials are *differentiable.*
- *Sums* of polynomials are polynomials.

$$\sum_{k=0}^{n} a_k x^k + \sum_{k=0}^{m} b_k x^k = \sum_{k=0}^{\max(n,m)} c_k x^k$$

- *Multiples* of a polynomials are polynomials.

$$c \sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} c\, a_k x^k$$

- *Products* of polynomials are polynomials.

$$\left(\sum_{k=0}^{n} a_k x^k \right) \cdot \left(\sum_{k=0}^{m} b_k x^k \right) = \sum_{k=0}^{n+m} c_k x^k$$

- *Compositions* of polynomials are polynomials.

$$\sum_{k=0}^{n} a_k \left(\sum_{l=0}^{m} b_l x^l \right)^k = \sum_{k=0}^{n \cdot m} c_k x^k$$

- *Derivatives* of polynomials are polynomials.

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n-1} c_k x^k$$

$\lhd$

10.2 Zeros

Theorem 10.3 (Number of zeros)**.** A polynomial $p : \mathbb{R} \to \mathbb{R}$ of order n has at most n zeros. $\lhd$

Example 10.2. Let $x \in \mathbb{R}$.

- Zeros of $x^2 - 1$: $\{-1, 1\}$
- Zeros of $x^2 + 1$: $\{\}$
- Zeros of $x^3 - x$: {-1,0,1}
- Zeros of $x^3 + x$: $\{0\}$

$\lhd$

Theorem 10.4 (Zeros of odd order polynomials)**.** An odd order polynomial $p : \mathbb{R} \to \mathbb{R}$ has at least one zero. $\lhd$

Theorem 10.5 (Separating zeros)**.** A polynomial p_n of n^{th} order with a zero x_0 may be expressed by

$$p_n(x) = (x - x_0) \cdot p_{n-1}(x)$$

where $p_{n-1}(x)$ is a polynomial of order $n - 1$ with

$$p_{n-1}(x) = \frac{p_n(x)}{x - x_0}$$

◁

Example 10.3. The polynomial

$$p_3 : \left\{ \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 - 3x^2 - x + 3 \end{array} \right.$$

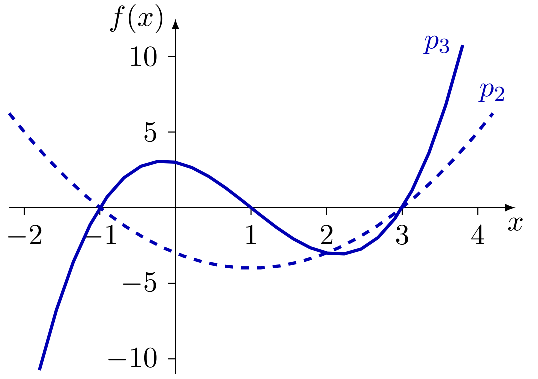
has (among other zeros) a zero at $x_0 = 1$. We divide the polynomial by this zero:

$$\begin{array}{l}
\;\;(x^3 - 3x^2 \quad - x + 3)/(x - 1) = x^2 - 2x - 3 \\
\underline{-(x^3 \;\; - x^2)} \\
\quad 0 \;\; - 2x^2 \;\; - x \\
\quad \underline{-(-2x^2 + 2x)} \\
\qquad\quad 0 \;\; - 3x + 3 \\
\qquad\quad \underline{-(\,-3x + 3)} \\
\qquad\qquad\qquad 0
\end{array}$$

Hence, we get the second order polynomial

$$p_2 : \left\{ \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 - 2x - 3 \end{array} \right.$$

with the two remaining zeros: $\{-1, 3\}$.



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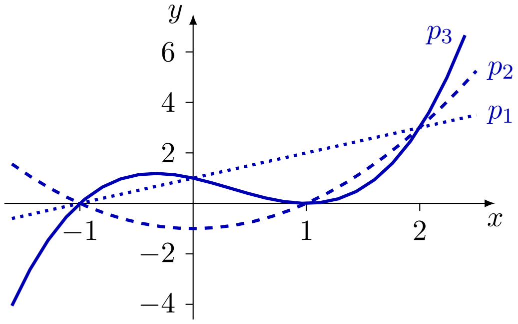
Definition 10.6 (Multiple zero)**.** If a polynomial $p : \mathbb{R} \to \mathbb{R}$ can be divided by a zero more than once without rest we call it a *multiple zero*. ◁

Example 10.4. The polynomial

$$p_3 : \left\{ \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 - x^2 - x + 1 \end{array} \right.$$

has a double zero at 1. I.e.

$$\begin{aligned}
p_3(x) &= x^3 - x^2 - x + 1 \\
p_2(x) &= \frac{p_3(x)}{x - 1} = x^2 - 1 \\
p_1(x) &= \frac{p_2(x)}{x - 1} = x + 1
\end{aligned}$$



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Theorem 10.7 (Fundamental theorem of algebra)**.** Every non-constant polynomial over complex numbers has at least one zero. ◁

Remark: The fundamental theorem of algebra includes complex zeros. Hence, polynomials like $p(x) = x^2 + 1$ have zeros in the complex domain.

If we divide a polynomial of order n by its zero $(x - x_0)$ we get a polynomial of order $n - 1$. If this new polynomial is of order greater 0 it again will have a zero. Hence, the fundamental theorem of algebra indirectly states that any polynomial of order n has exactly n zeros.

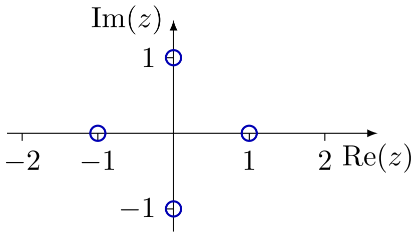
Example 10.5. The polynomial

$$f : \left\{ \begin{array}{l} \mathbb{C} \to \mathbb{C} \\ z \mapsto z^4 - 1 \end{array} \right.$$

has four zeros:

$$\begin{aligned}
f(z) &= z^4 - 1 = (z^2 - 1)(z^2 + 1) \\
&= (z - 1)(z + 1)(z - \mathrm{j})(z + \mathrm{j})
\end{aligned}$$

i.e. $z_1 = 1$, $z_2 = -1$, $z_3 = \mathrm{j}$ and $z_4 = -\mathrm{j}$.



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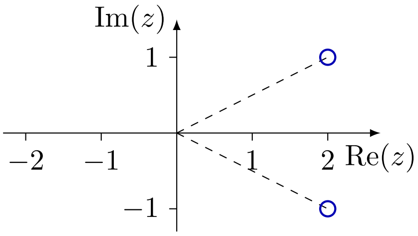
Theorem 10.8 (Conjugate complex zeros)**.** The complex zeros of a polynomial $p : \mathbb{C} \to \mathbb{C}$ with real coefficients appear in complex conjugate pairs. I.e. for every complex zero there exist a conjugate zero. ◁

Example 10.6. The polynomial

$$f : \begin{cases} \mathbb{C} \to \mathbb{C} \\ z \mapsto z^2 - 4z + 5 \end{cases}$$

has two zeros at $z_{1,2} = 2 \pm \mathrm{j}$:

$$\begin{aligned} p_2(z) &= z^2 - 4z + 5 \\ p_1(z) &= \frac{p_2(z)}{z - z_1} = \frac{z^2 - 4z + 5}{z - (2 + \mathrm{j})} = z - 2 + \mathrm{j} \\ p_0 &= \frac{p_1(z)}{z - z_2} = \frac{z - 2 + \mathrm{j}}{z - (2 - \mathrm{j})} = 1 \end{aligned}$$



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Remark: When separating a polynomial into its zeros, the complex conjugate zeros may be kept together resulting in real coefficients. I.e. with complex conjugate zeros at $z_{1,2} = a \pm \mathrm{j}b$ we have

$$\begin{aligned} (z - z_1)(z - z_2) &= (z - (a + b\mathrm{j}))(z - (a - b\mathrm{j})) \\ &= z^2 - 2ax + a^2 + b^2 \\ &= z^2 + Az + B \end{aligned}$$

with

$$A = -2a \qquad B = a^2 + b^2$$

$$a = -\frac{A}{2} \qquad b = \pm\sqrt{B - \frac{A^2}{4}}$$

10.3 Rational function

Definition 10.9 (Rational function)**.** Let p and q be polynomials and $q \neq 0$. We call

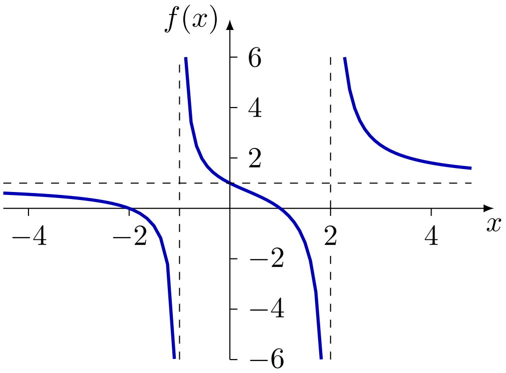
$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \frac{p(x)}{q(x)} \end{cases}$$

a *rational function.* ◁

Example 10.7. Let

$$f : \begin{cases} \mathbb{R} \setminus \{-1, 2\} \to \mathbb{R} \\ x \mapsto \dfrac{x^2 + x - 2}{x^2 - x - 2} \end{cases}$$

be a rational function. Since the denominator has two zeros at -1 and 2 we exclude these values from the domain.



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10.4 Properties of rational functions

Theorem 10.10 (Zeros of rational functions)**.** A rational function has a zero where the numerator has a zero and the denominator is not zero. ◁

Definition 10.11 (Pole)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a rational function. We call a point x_0 with the property

$$\lim_{x \to x_0} |f(x)| = \infty$$

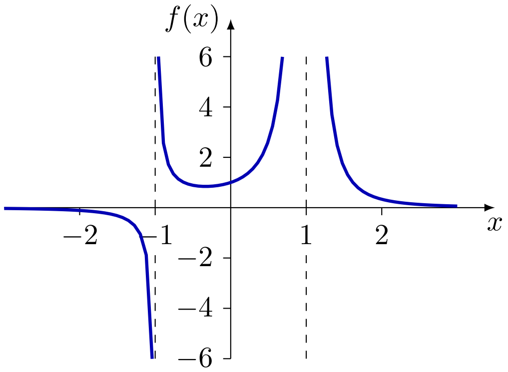
a *pole.* ◁

Remark: When taking the left- and right-side limit there are poles with equal and different signs for infinity.

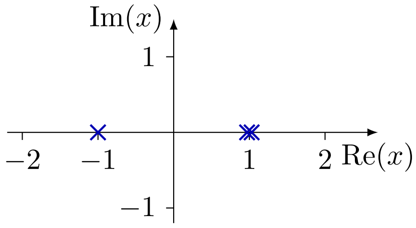
Example 10.8. The rational function

$$f : \begin{cases} \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R} \\ x \mapsto \dfrac{1}{x^3 - x^2 - x + 1} \end{cases}$$

has a pole at -1 with changing signs and a second pole at 1 with equal signs.



Although the domain of f is real only we plot the poles already on the complex plane of x. We do this to be prepared for complex poles and zeros.



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Theorem 10.12 (Pole of a rational function)**.** A rational function has a pole at x_0 where the denominator has a zero and the numerator is not zero. ◁

Remark: What is the situation if both, numerator and denominator are zero? We need l'Hôpital's rule:

Theorem 10.13 (l'Hôpital's rule)**.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions, $x_0 \in \mathbb{R}$, $c \in \mathbb{R} \cup \{-\infty, +\infty\}$ with

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \text{ or } \pm\infty$$

and

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = c$$

then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = c$$

◁

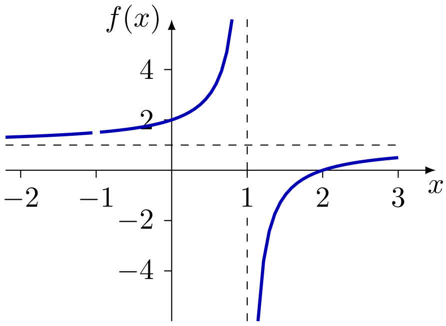
Example 10.9. The rational function

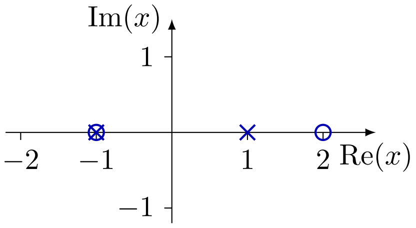
$$f : \begin{cases} \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R} \\ x \mapsto \dfrac{x^2 - x - 2}{x^2 - 1} \end{cases}$$

has in its numerator zeros at -1 and 2 and in its denominator at -1 and 1. Hence, f has a zero at 2 and a pole at 1. For the common zero of numerator and denominator we apply l'Hôpital's rule:

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{2x - 1}{2x} = \frac{2(-1) - 1}{2(-1)} = \frac{3}{2}$$

I.e. for $x_0 = -1$ the rational function f approaches $\frac{3}{2}$, however, since -1 does not belong to the domain f will never have this value.





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Remark: A rational function is discontinuous at points where the denominator is zero. However, if the left- and right-side limits exist and are equal we call it a *removable discontinuity*.

10.5 Partial fraction decomposition

Rational functions with large order polynomials in numerator and denominator are difficult to handle as they are. We are looking for a technique to split them into a sum of more basic fractions.

The technique of *partial faction decomposition* changes a rational function into a sum of low-order rational functions.

We need this e.g. to perform integration or Laplace-transforms.

10.5.1 Order of numerator and denominator

In a first step we need to assure that the order of the numerator polynomial is less than the order of the denominator polynomial. If this is not the case, we perform a polynomial division where the rest is a fraction with this requirement.

Example 10.10. For the rational function

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{x^3 + 3x^2 - x - 3}{x^2 - 4} \end{cases}$$

the order of the numerator polynomial is not less than the one of the denominator polynomial. Hence we perform a polynomial division:

$$\begin{array}{l} (x^3 + 3x^2 \ - x \ \ - 3)/(x^2 - 4) = x + 3 + \frac{3x+9}{x^2-4} \\ \underline{-(x^3 \qquad\quad - 4x)} \\ \qquad\quad +3x^2 + 3x \ \ - 3 \\ \quad \underline{-(+3x^2 \qquad\quad - 12)} \\ \qquad\qquad\quad +3x \ \ + 9 \end{array}$$

Hence we have:

$$f(x) = \frac{x^3 + 3x^2 - x - 3}{x^2 - 4} = x + 3 + \frac{3x + 9}{x^2 - 4}$$

◁

For the partial fraction decomposition we concentrate on the fraction only.

10.5.2 Partial fractions for real poles

Let us assume a rational function f with the order n of the polynomial of its denominator being greater than the one of its numerator.

If the polynomial of the denominator has real zeros only we create the partial fractions as follows:

- for a single (not multiple) zero x_0:

$$\frac{A}{x - x_0}$$

- for a double zero x_0:

$$\frac{A_1}{x - x_0} + \frac{A_2}{(x - x_0)^2}$$

- for an n-multiple zero x_0:

$$\frac{A_1}{x - x_0} + \frac{A_2}{(x - x_0)^2} + \ldots + \frac{A_n}{(x - x_0)^n}$$

We set the sum of these fractions equal to f, multiply by the denominator of f and sort the summands my the powers of x. Now we compare the coefficients of the powers of x on both sides to evaluate the constants of the partial fractions.

Example 10.11. We take the remainder of the previous example:

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{3x + 9}{x^2 - 4} \end{cases}$$

We have two real zeros $z_1 = 2$ and $z_2 = -2$ and we get:

$$f(x) = \frac{3x + 9}{x^2 - 4} = \frac{A}{x - 2} + \frac{B}{x + 2}$$

We multiply with the denominator of $f(x)$:

$$\begin{aligned} 3x + 9 &= \frac{A(x^2 - 4)}{x - 2} + \frac{B(x^2 - 4)}{x + 2} \\ &= A(x + 2) + B(x - 2) \\ &= (A + B)x + 2(A - B) \end{aligned}$$

Comparing the coefficients we get the two equations:

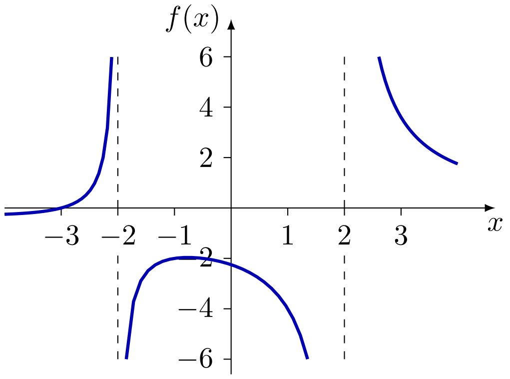
$$3 = A + B \qquad \text{and} \qquad 9 = 2A - 2B$$

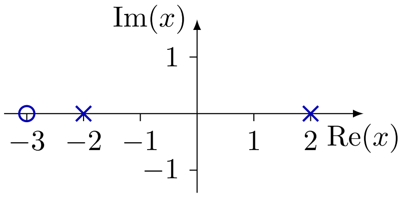
Solving these two equations for A and B gives:

$$A = \frac{15}{4} \qquad \text{and} \qquad B = -\frac{3}{4}$$

Hence we get:

$$f(x) = \frac{3x + 9}{x^2 - 4} = \frac{\frac{15}{4}}{x - 2} - \frac{\frac{3}{4}}{x + 2}$$





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Example 10.12. The rational function

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{x^2 + 2x + 1}{x^3 - 3x + 2} \end{cases}$$

has in its denominator a single zero at -2 and a double zero at 1. Hence we separate f into three partial fractions:

$$\begin{aligned}
\frac{x^2+2x+1}{x^3-3x+2} &= \frac{A}{x+2} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2} \\
x^2+2x+1 &= (x-1)^2A + (x-1)(x+2)B_1 \\
&\quad + (x+2)B_2 \\
&= (A+B_1)x^2 + (B_1+B_2-2A)x \\
&\quad + (A-2B_1+2B_2)
\end{aligned}$$

Comparing the three coefficients we get:

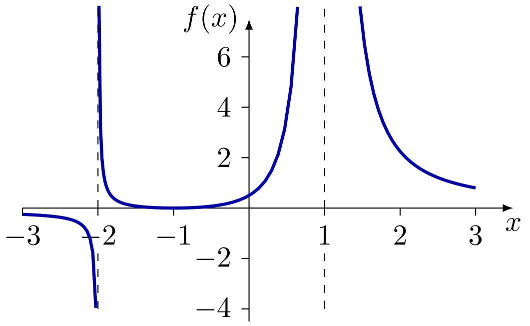
$$\begin{aligned}
1 &= A + B_1 \\
2 &= B_1 + B_2 - 2A \\
1 &= A - 2B_1 + 2B_2
\end{aligned}$$

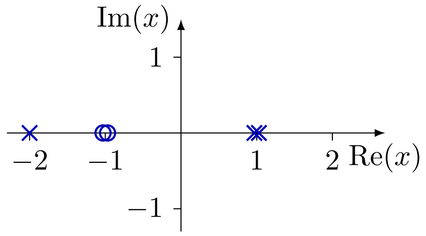
Solving these three equations gives:

$$A = \frac{1}{9}, \qquad B_1 = \frac{8}{9} \qquad \text{and} \qquad B_2 = -\frac{4}{3}$$

and we get:

$$f(x) = \frac{x^2+2x+1}{x^3-3x+2} = \frac{\frac{1}{9}}{x+2} + \frac{\frac{8}{9}}{x-1} + \frac{\frac{4}{3}}{(x-1)^2}$$





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10.5.3 Partial fractions for complex poles

Let us assume a rational function f with the order of the polynomial of its denominator being greater than the one of its numerator.

This time we assume that the denominator polynomial also includes complex zeros. For real coefficients every complex zero z_1 results a second complex conjugate zero $\overline{z_1}$. We combine the complex conjugate zeros:

$$\begin{aligned}
(x-z_1)(x-\overline{z_1}) &= x^2 - (z_1+\overline{z_1})x + z_1\overline{z_1} \\
&= x^2 + ax + b
\end{aligned}$$

with

$$a = -(z_1+\overline{z_1}) \qquad \text{and} \qquad b = z_1\overline{z_1}$$

We use this term as denominator for partial fractions:

- for a single (not multiple) pair of zeros:
$$\frac{Ax+B}{x^2+ax+b}$$
- for a double pair of zeros:
$$\frac{A_1x+B_1}{x^2+ax+b} + \frac{A_2x+B_2}{(x^2+ax+b)^2}$$
- for an n-multiple pair of zeros:
$$\frac{A_1x+B_1}{x^2+ax+b} + \ldots + \frac{A_nx+B_n}{(x^2+ax+b)^n}$$

Example 10.13. The denominator of the rational function

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{x^2-4}{x^3-x^2+2} \end{cases}$$

has a real zero at -1 and a complex conjugate zero at $1 \pm \mathrm{j}$. Hence we separate f into two partial fractions:

$$\begin{aligned}
\frac{x^2-4}{x^3-x^2+2} &= \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+2} \\
x^2-4 &= (x^2-2x+2)A \\
&\quad + (x+1)(Bx+C) \\
&= (A+B)x^2 + (B+C-2A)x \\
&\quad + (2A+C)
\end{aligned}$$

Comparing the three coefficients we get:

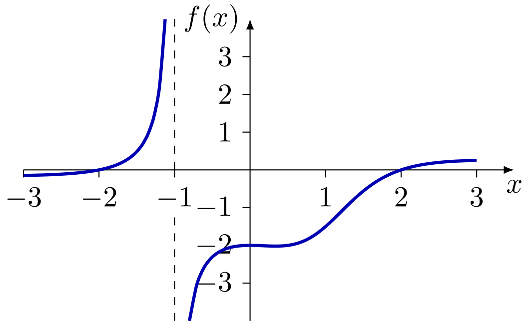
$$\begin{aligned}
1 &= A + B \\
0 &= B + C - 2A \\
-4 &= 2A + C
\end{aligned}$$

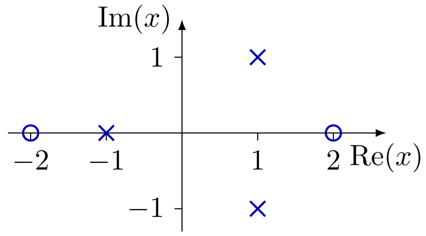
Solving these three equations gives:

$$A = -\frac{3}{5}, \qquad B = \frac{8}{5} \qquad \text{and} \qquad C = -\frac{14}{5}$$

and we get:

$$f(x) = \frac{x^2-4}{x^3-x^2+2} = \frac{-\frac{3}{5}}{x+1} + \frac{\frac{8}{5}x - \frac{14}{5}}{x^2-2x+2}$$





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10.6 Taylor polynomial

Polynomials are readily to handle e.g. for differentiation or, as we see later, for integration. Hence, if it is possible to approximate functions by polynomials, further calculations become easier.

If we want to approximate a function f at point x_0 by a zero order polynomial (i.e. a constant) we simply take the value $f(x_0)$ as the coefficient a_0 for the polynomial p_0:

$$p_0 = a_0 = f(x_0)$$

To improve the approximation we may want to adjust the slope of f at x_0 to a first order polynomial. I.e. the first derivative of function and polynomial shall be equal and we get the equations:

$$\begin{aligned} p_1(x_0) &= f(x_0) \\ p_1'(x_0) &= f'(x_0) \end{aligned}$$

In order to further improve the approximation we may include higher order derivatives:

$$\begin{aligned} p_n(x_0) &= f(x_0) \\ p_n'(x_0) &= f'(x_0) \\ p_n''(x_0) &= f''(x_0) \\ \vdots \quad &\quad \vdots \\ p_n^{(n)}(x_0) &= f^{(n)}(x_0) \end{aligned}$$

The resulting polynomial is the *Taylor polynomial*:

Definition 10.14 (Taylor polynomial)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be an n times differentiable function. We define the *Taylor polynomial* as

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

with $x_0 \in \mathbb{R}$ as the *expansion point*. ◁

Example 10.14. We want to approximate the function

$$f : \begin{cases} \mathbb{R}_{>-1} \to \mathbb{R} \\ x \mapsto \frac{1}{x+1} \end{cases}$$

by Taylor polynomials at expansion point $x_0 = 0$. The derivatives of f are:

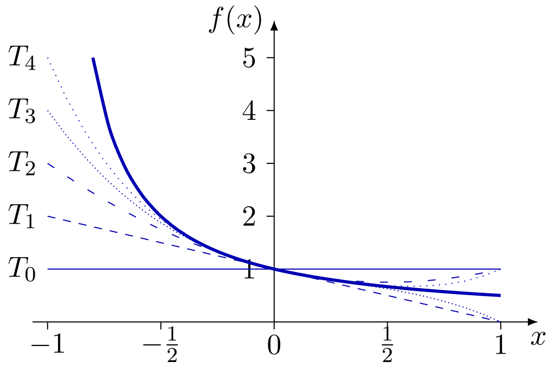
$$\begin{aligned} f'(x) &= \frac{-1}{(x+1)^2} \\ f''(x) &= \frac{2}{(x+1)^3} \\ f'''(x) &= \frac{-6}{(x+1)^4} \\ \vdots \quad &\quad \vdots \\ f^{(n)}(x) &= \frac{(-1)^n n!}{(x+1)^{n+1}} \end{aligned}$$

With $x_0 = 0$ we get

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \\ &= \sum_{k=0}^{n} \frac{(-1)^k}{(x_0+1)^{k+1}}(x - x_0)^k \\ &= \sum_{k=0}^{n} (-x)^k \end{aligned}$$

I.e.

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= 1 - x \\ T_2(x) &= 1 - x + x^2 \\ T_3(x) &= 1 - x + x^2 - x^3 \\ T_4(x) &= 1 - x + x^2 - x^3 + x^4 \\ \vdots \quad &\quad \vdots \end{aligned}$$



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An application of the Taylor polynomial is the *propagation of uncertainty*. In engineering we perform measurements with an uncertainty due to e.g. tolerances of the equipment. See the following example:

Example 10.15. A car with mass of 1000 kg drives with a speed of 90 km/h = 25 m/s. The speed has been measured with an accuracy of 1%, i.e.

$$v = 25\,\mathrm{m/s} \qquad \Delta v = 0.25\,\mathrm{m/s}$$

We want to estimate the kinetic energy of the car which is given by

$$\begin{aligned} E_{\text{kin}} &= f(v) = \frac{m}{2}v^2 \\ &= \frac{1\,000\,\mathrm{kg}}{2}(25\,\mathrm{m/s})^2 = 312.5\,\mathrm{kJ} \end{aligned}$$

The expected uncertainty of the kinetic energy can be estimated by the Taylor polynomial:

$$\begin{aligned} T_n(v+\Delta v) &= \sum_{k=0}^{n} \frac{f^{(n)}(v)}{k!}((v+\Delta v)-v)^k \\ &= f(v) + f'(v)\Delta v + \tfrac{f''(v)}{2}\Delta v^2 + \dots \end{aligned}$$

Since Δv is sufficient small we may stop after the linear summand, i.e.

$$T_1(v+\Delta v) = f(v) + f'(v)\Delta v$$

The first summand is the function itself whereas the second summand is the propagation of uncertainty. In our case:

$$\begin{aligned} \Delta E_{\text{kin}} &\approx \tfrac{\mathrm{d}}{\mathrm{d}v}E_{\text{kin}}(v)\Delta v = \tfrac{\mathrm{d}}{\mathrm{d}v}\tfrac{m}{2}v^2\Delta v = mv\Delta v \\ &= 1\,000\,\mathrm{kg}\cdot 25\,\mathrm{m/s}\cdot 0.25\,\mathrm{m/s} = 6.25\,\mathrm{kJ} \end{aligned}$$

which is 2% of the kinetic energy. ◁

10.7 Problems

Problem 10.1: Which of the following functions $f : \mathbb{R} \to \mathbb{R}$ are polynomials? If so, what is the order?

1. $f(x) = 1 + x + x^2 + x^3 + x^4$
2. $f(x) = x^3 - 3$
3. $f(x) = 0$
4. $f(x) = \sqrt{3x^4} - x + 1$
5. $f(y) = \sqrt{x}$
6. $f(z) = z^{100}$

Problem 10.2: Let p_1 and p_2 be polynomials of 3rd and 4th order, respectively. Which of the following terms are polynomials? If so, what is the order?

1. $p_1 + p_2$
2. $p_1 \cdot p_2$
3. $p_1' + 2p_2$
4. $3p_1 + 2p_2'$

Problem 10.3: Write the following polynomials as the product of their zeros.

1. $p_1(x) = x^2 - 1$
2. $p_2(x) = 2x^3 - 2x^2 - 4x$
3. $p_3(x) = x^2 - (a+b)x + ab$
4. $p_4(z) = z^2 - 2xz + x^2$
5. $p_5(x) = x^2 + 2x + 2$
6. $p_6(x) = x^3 - 3x^2 + 7x - 5$

Problem 10.4: Perform polynomial division:

1. $(x^3 - 6x^2 - 11x - 6)/(x - 3)$
2. $(x^3 - 6x^2 - 11x - 6)/(x - 2)$
3. $(x^3 - 6x^2 - 11x - 6)/(x - 1)$
4. $(x^4 - 5x^2 + 4)/(x^2 - x - 2)$

Problem 10.5: For a polynomial with real coefficients a complex zero z_1 is known. What can you say about other zeros and what is the minimum order of the polynomial?

Problem 10.6: Plot the following function:

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{x^2 - 1}{x^2 - 4} \end{cases}$$

Problem 10.7: Analyse the following functions with respect to zeros and poles.

1. function of previous problem.
2. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{1}{x^2 + x - 2} \end{cases}$
3. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{x^2 - 1}{x^2 - 4x + 4} \end{cases}$
4. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{x^2 - x - 2}{x^2 - 4} \end{cases}$

Problem 10.8: Create a rational function $f : \mathbb{R} \to \mathbb{R}$ with zeros at ± 1, poles at ± 3.

Problem 10.9: Separate from the following functions $f : \mathbb{R} \to \mathbb{R}$ a polynomial in order that the remaining rational function has a higher order in its denominator.

1. $f(x) = \dfrac{x^4}{x^2 - 1}$
2. $f(x) = \dfrac{x^4 - 2x^2 + 1}{x^2 + x - 2}$
3. $f(x) = \dfrac{x^6}{x^3 - x^2 + x - 1}$
4. $f(x) = \dfrac{x^5 - x^4 + x^3}{x^2 - x + 1}$

Problem 10.10: Perform partial fraction decomposition for the functions $f : \mathbb{R} \to \mathbb{R}$.

1. $f(x) = \dfrac{1}{x^2 - 1}$
2. $f(x) = \dfrac{2x + 3}{x^2 + 4x + 4}$
3. $f(x) = \dfrac{x^2 + 1}{x^3 + 2x^2 + 2x}$
4. $f(x) = \dfrac{x^3 + x}{x^4 + 2x^3 + x^2 - 2x - 2}$

Problem 10.11: Evaluate the 2^{nd} order Taylor polynomial at expansion point $x_0 = 0$ of the exponential function.

Problem 10.12: Evaluate the 1^{st} order Taylor polynomial at expansion point $x_0 = 1$ of the natural logarithm function.

11 Curve sketching

11.1 Introduction

Curve sketching is about analysing a function with respect to a number of aspects. We ask question about

- domain, codomain and boundary
- monotonicity, symmetry and periodicity
- intersection with axes
- extrema
- curvature
- discontinuities
- asymptotes
- drawing a sketch

Once you read the lecture notes until here, most of these issues are known already. However, we will look at some of them from a slightly different point of view.

If you are asked to perform a curve sketching for a function f, you give a statement on all these properties listed above.

11.2 Domain, codomain and boundary

When investigating a function f you first should look at the *domain* (range of definition), the *codomain* and the *image* of f.

The *domain* is the set X of all values $x \in X$ for which $f(x)$ exist.

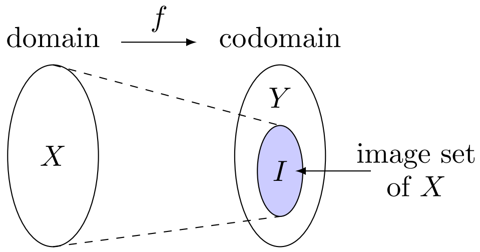
$$X = \{x \mid \forall x \; \exists f(x)\}$$

The *image* I is the set of all values of f

$$I = f(X)$$

The *codomain* Y includes all elements of the image I. However, there may be elements $y \in Y$ which are not elements of the image $y \notin I$. I.e.

$$f(X) = I \subseteq Y$$



If the absolute of all elements of the image are smaller than a given limit $M \in \mathbb{R}$ we say the function f is *bounded*.

$$|f(x)| < M \in \mathbb{R} \qquad \text{for all } x \in X$$

11.3 Monotonicity, symmetry and periodicity

With the earlier definition of monotonicity and the first derivative of a function we derive the following theorem:

Theorem 11.1 (Derivatives and monotonicity)**.** Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be a differentiable function. The function is

- *increasing* if $f'(x) \geq 0$
- *decreasing* if $f'(x) \leq 0$

for all $x \in I$. The function is *strictly* increasing (or decreasing) if there exist only discrete $x \in I$ where the first derivative is zero. ◁

We introduced two types of symmetry: reflection symmetry and point symmetry.

Definition 11.2 (Symmetry)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

- If there exist a value $x_0 \in \mathbb{R}$ with

  $$f(x_0 - x) = f(x_0 + x)$$

  we say f has *reflection symmetry* at x_0.

- If there exist $x_0, y_0 \in \mathbb{R}$ with

  $$f(x_0 - x) - y_0 = y_0 - f(x_0 + x)$$

  we say f has *point symmetry* with respect to point (x_0, y_0).

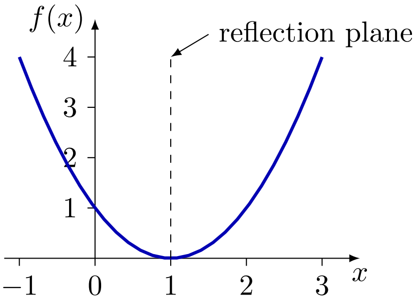
◁

Example 11.1.

- The function

$$f: \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto (x-1)^2 = x^2 - 2x + 1 \end{cases}$$

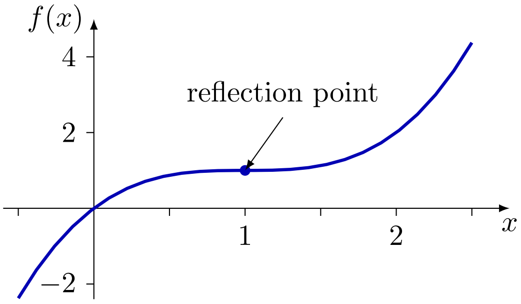
shows reflection symmetry at $x_0 = 1$.



- The function

$$f: \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto (x-1)^3 + 1 = x^3 - 3x^2 + 3x \end{cases}$$

shows point symmetry at $(1,1)$.



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Definition 11.3 (Periodicity)**.** A function $f: \mathbb{R} \to \mathbb{R}$ is *periodic* if there exist a $T \in \mathbb{R}$ such that

$$f(x) = f(x + nT) \qquad \text{for all } n \in \mathbb{Z}$$

◁

11.4 Intersection with axes

We analyse the intersections of the investigated function $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$ with ordinate and abscissa of the Cartesian coordinate system.

The intersection point with the ordinate is trivial:

$$(0, f(0))$$

If zero is not an element of the domain, $0 \notin D$, the intersection point does not exist.

The intersection points with the abscissa are the zeros of the function. With n as the number of zeros and x_k, $k = 1, \ldots, n$ as the zeros of f the intersection points with the abscissa are

$$(x_k, 0) \qquad k = 1, \ldots, n$$

For polynomials there exist analytical methods to find zeros up to the order of four. If a zero has been found, it can be separated from the polynomial by polynomial division. This reduces the order of the polynomial and the number of remaining zeros.

11.5 Extrema

Theorem 11.4 (Extrema and derivatives)**.** If a differentiable function $f: \mathbb{R} \to \mathbb{R}$ has an extremum at $x_0 \in \mathbb{R}$ then x_0 is a zero of the first derivative of f, $f'(x_0) = 0$. I.e. the tangent of the function at x_0 is horizontal. ◁

Remark: This implication means that a zero of the first derivative is necessary condition for an extremum. However, not every zero of the first derivative results in an extremum of the function.

E.g. the function $f: \mathbb{R} \to \mathbb{R}, x \mapsto x^3$ has no extremum at $x_0 = 0$ although it is a zero of the first derivative of f.

Hence, a zero for the first derivative is a necessary but not sufficient condition for an extremum.

Theorem 11.5 (Derivatives and extrema)**.** Let $f: \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. If there exist a number $x_0 \in \mathbb{R}$ where the first derivative is zero and the second derivative is not zero then f has an extremum at x_0. If $f''(x_0) < 0$ then it is a maximum, if $f''(x_0) > 0$ then it is a minimum. ◁

Remark: Again, this is an implication an does not hold the other way round. E.g. the function $f: \mathbb{R} \to \mathbb{R}, x \mapsto x^4$ has an extremum at $x_0 = 0$ but both, the first and second derivative are zero at x_0.

A general approach is to differentiate a function f at x_0 until we find for the first time a derivative with $f^{(n)}(x_0) \neq 0$. If $n \in \mathbb{N}$ is even, x_0 is an extremum of f. If n is odd, x_0 is no extremum of f.

Another general approach is to study the first derivative in more detail. If the first derivative f' of a differentiable function $f : \mathbb{R} \to \mathbb{R}$ has a zero x_0 and it crosses the abscissa at this point, then f has an extremum at x_0. If the first derivative just touches the abscissa at x_0 without crossing it, f has no extremum at x_0.

Theorem 11.6 (Condition for extrema)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. A necessary and sufficient condition for an extremum at x_0 is that the first derivative of f at x_0 is zero and that it crosses the abscissa at x_0. ◁

Remark: If a function is defined over a closed interval, an extremum can be at the end of the interval, see the following example.

Example 11.2.

- The function

$$f : \begin{cases} [0,3] \to \mathbb{R} \\ x \mapsto x^2 \end{cases}$$

has a minimum at 0 and a maximum at 3.

- The function

$$f : \begin{cases} (0,3) \to \mathbb{R} \\ x \mapsto x^2 \end{cases}$$

has no extrema.

◁

11.6 Curvature

If the second derivative f'' of a function $f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x)$ is not equal to zero, the function is curved, i.e. the slope of f changes over x.

If the second derivative is positive the function describes a left curve, i.e. for increasing x the slope increases.

If the second derivative is negative the function describes a right curve, i.e. for increasing x the slope of f decreases.

Definition 11.7 (Convex and concave functions)**.** Let $f : I \to \mathbb{R}$ be a twice differentiable function and $I \subseteq \mathbb{R}$ an interval.

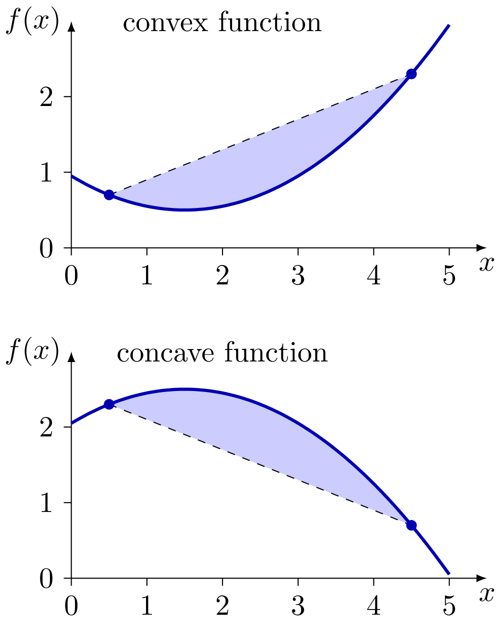
- We say f is a *convex function* if

$$f''(x) > 0 \qquad \text{for all } x \in I$$

- We say f is a *concave function* if

$$f''(x) < 0 \qquad \text{for all } x \in I$$

◁



Of interest are the points, where a function changes its direction:

Theorem 11.8 (Inflection point and derivatives)**.** If a twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ has an *inflection point* at $x_0 \in \mathbb{R}$ then x_0 is a zero of the second derivative of f, $f''(x_0) = 0$. I.e. the tangent of the first derivative at x_0 is horizontal. ◁

Theorem 11.9 (Derivatives and inflections)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a three times differentiable function. If there exist a number $x_0 \in \mathbb{R}$ where the second derivative is zero and the third derivative is not zero then f has an inflection point at x_0. If $f'''(x_0) < 0$ then the left curve changes to right curve, if $f'''(x_0) > 0$ then the right curve changes to a left curve. ◁

Remark: There are situations where the second *and* the third derivatives are zero. There are general approaches to find inflection points:

- If the second derivative of a function f has a zero x_0 we differentiate over and over again until we find a derivative with $f^{(n)}(x_0) \neq 0$. If $n \in \mathbb{N}$ is odd, f has an inflection point at x_0. If n is even, f has no inflection point at x_0.

- If the second derivative of a function f has a zero at x_0 we study this second derivative in more detail. If f'' crosses the abscissa at x_0, then f has an inflection point at x_0. If f'' just touches the abscissa at x_0 without crossing it, f has no inflection point at x_0.

Definition 11.10 (Saddle point)**.** We call an inflection point with zero slope a *saddle point*. $\triangleleft$

Example 11.3.

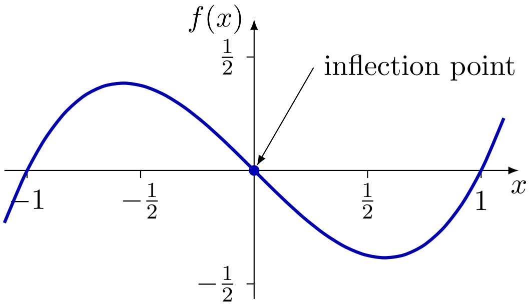
- The function

  $$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 - x \end{cases}$$

  has the derivatives

  $$f'(x) = 3x^2 - 1, \quad f''(x) = 6x, \quad f'''(x) = 6$$

  At $x_0 = 0$ the second derivative is zero whereas the third derivative is 6. Hence f has an inflection point at 0 where a right curve changes to a left curve.



- The function

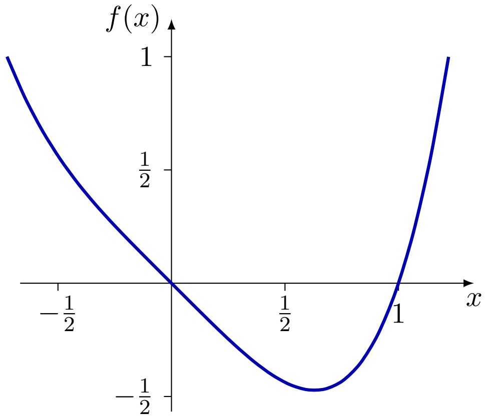
  $$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^4 - x \end{cases}$$

  has the derivatives

  $$f'(x) = 4x^3 - 1, \qquad f''(x) = 12x^2,$$
  $$f'''(x) = 24x \qquad f^{(4)}(x) = 24$$

  At $x_0 = 0$ the second derivative is zero. However, it turns out that the third derivative has also a zero at $x_0 = 0$ and the fourth derivative has the value 24. Hence f has no inflection point.

  This becomes also clear by investigating the second derivative. $f''(x) = 12x^2$ touches the abscissa at $x_0 = 0$ but does not cross it. Hence, there can not be an inflection point at $x_0 = 0$.



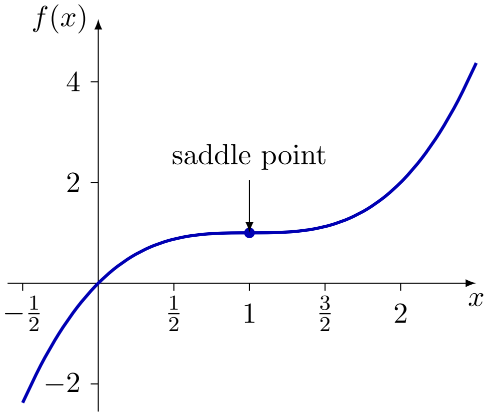
- The function

  $$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 - 3x^2 + 3x \end{cases}$$

  has the derivatives

  $$f'(x) = 3x^2 - 6x + 3,$$
  $$f''(x) = 6x - 6, \qquad f'''(x) = 6$$

  At $x_0 = 1$ the second derivative is zero and the third derivative is 6. Hence f has an inflection point at $x_0 = 1$ where a right curve changes to a left curve. Since the first derivative is also zero at $x_0 = 1$ the inflection point is a saddle point.



$\triangleleft$

11.7 Discontinuities

A *discontinuity* is a point where a function is not continuous. We recall the condition for continuity at position x_0:

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$$

If the left- and right-side limit approaches $\pm\infty$ we say it is a *pole*.

If both, the left- and right-side limit exist and they are different we say it is a *jump discontinuity*.

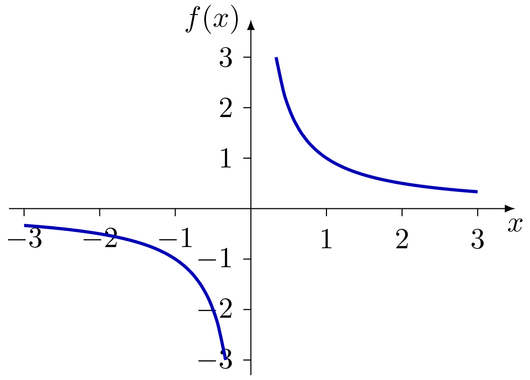
If both, the left- and right-side limit exist and they are equal but the function value does not exist we say it is a *removable discontinuity*.

Example 11.4.

- The function

$$f : \begin{cases} \mathbb{R} \setminus \{0\} \to \mathbb{R} \\ x \mapsto \dfrac{1}{x} \end{cases}$$

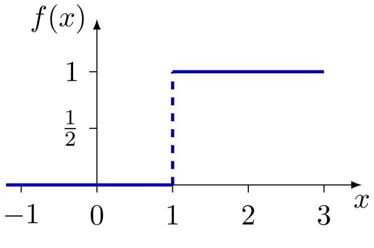
has a pole at $x_0 = 0$.



- The function

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto 0 \quad \text{for } x \le 1 \\ x \mapsto 1 \quad \text{for } x > 1 \end{cases}$$

has a jump discontinuity at $x_0 = 1$. (This would also be the case if f is not defined at $x_0 = 1$.)

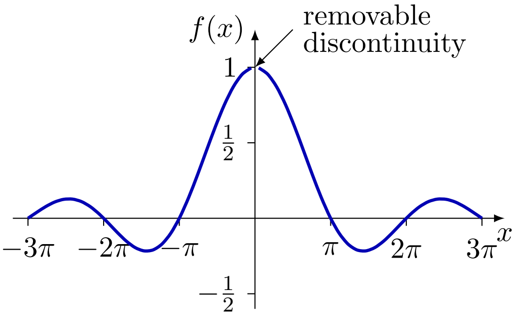


- The function

$$\operatorname{sinc} : \begin{cases} \mathbb{R} \setminus \{0\} \to \mathbb{R} \\ x \mapsto \dfrac{\sin(x)}{x} \end{cases}$$

has a removable discontinuity at $x_0 = 0$. We investigate the left- and right-side limit at once by applying l'Hôpitals's rule:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1$$



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11.8 Asymptotes

An *asymptote* is a simplified function to which an investigated function approaches arbitrary close towards infinity. There are different types of asymptotes:

A *vertical asymptote* appears at poles of a function. I.e. the asymptote is a straight vertical line at a given value x_0.

A function has a *horizontal asymptote* if its value approaches a constant value:

$$\lim_{x\to\infty} f(x) = c$$

E.g. all rational functions with an order of the denominator polynomial larger than the numerator polynomial approach zero.

Finally the asymptote may be a *simplified function*. E.g. If the order of the numerator polynomial of a rational function is not less than than the order of its denominator polynomial we may find the asymptote by polynomial division and neglecting the remaining fraction, see examples below.

Example 11.5.

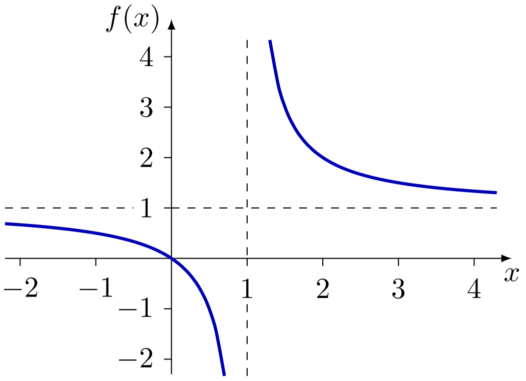
- The function

$$f : \begin{cases} \mathbb{R} \setminus \{1\} \to \mathbb{R} \\ x \mapsto \dfrac{x}{x-1} \end{cases}$$

has a pole at $x_0 = 1$ and approaches 1 towards ±∞. Hence we have two asymptotes: one at $y = 1$, the other at $x = 1$.

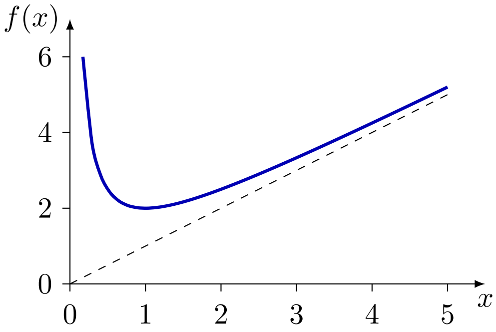
The limits at the pole are

$$\lim_{x\to 1^-} \frac{x}{x-1} = -\infty, \qquad \lim_{x\to 1^+} \frac{x}{x-1} = +\infty$$

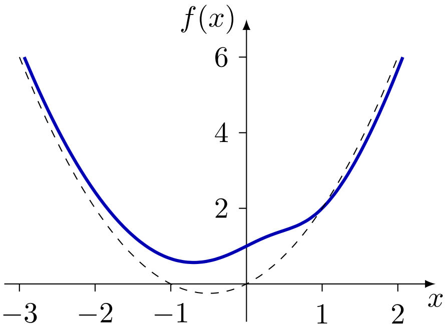


- For the function
$$f:\begin{cases}\mathbb{R}_{>0}\to\mathbb{R}\\ x\mapsto \dfrac{x^2+1}{x}\end{cases}$$
has a vertical asympotote at $x=0$ and a linear asymptote with $y=x$.

  The right-side limit of the pole is $+\infty$.
$$\lim_{x\to 0^+}\frac{x^2+1}{x}=\infty$$



- For the function
$$f:\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\mapsto \dfrac{x^4-1}{x^2-x+1}\end{cases}$$
we perform polynomial division and find:
$$f(x)=\frac{x^4-1}{x^2-x+1}=x^2+x-\frac{x-1}{x^2-x+1}$$
Hence, the asymptote is $y=x^2+x$.



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11.9 Drawing a sketch

Finally, with the knowledge of all the discussed properties of a function it is useful to draw a sketch of the function.

The sketch should visualize all zeros, extrema, poles, inflection points and asymptotes. Hence, the range should be chosen as such that all these elements are visible.

As examples have a look at the sketches of this chapter.

11.10 Problems

Problem 11.1: Find the image of the following functions.

1. $f:\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\mapsto x^2-1\end{cases}$
2. $f:\begin{cases}\mathbb{R}\setminus\{0\}\to\mathbb{R}\\ x\mapsto x^{-1}\end{cases}$
3. $f:\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\mapsto \sin(3x)-3\end{cases}$
4. $f:\begin{cases}\mathbb{R}\setminus\{0\}\to\mathbb{R}\\ x\mapsto x^4+1\end{cases}$

Problem 11.2: Analyse the following functions with respect to monotonicity, symmetry and periodicity.

1. $f:\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\mapsto x^3+x\end{cases}$
2. $f:\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\mapsto x^3-x\end{cases}$
3. $f:\begin{cases}\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to\mathbb{R}\\ x\mapsto \tan(x)\end{cases}$
4. $f:\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\mapsto \exp(x^2)\end{cases}$
5. $f:\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\mapsto x^2-2x+1\end{cases}$
6. $f:\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\mapsto \exp(x)\sin(x)\end{cases}$
7. $f:\begin{cases}\mathbb{R}\setminus\{x\mid x=n\pi, n\in\mathbb{Z}\}\to\mathbb{R}\\ x\mapsto \cot(x)\end{cases}$
8. $f:\begin{cases}\mathbb{R}\to\mathbb{R}\\ x\mapsto \cos(2x)\end{cases}$

Problem 11.3: Find the intersections with abscissa and ordinate.

1. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 - x \end{cases}$
2. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 - x + 1 \end{cases}$
3. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \sin(x) \end{cases}$
4. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto e^x - e \end{cases}$

Problem 11.4: Find x- and y-coordinates of all extrema.

1. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 - 2x + 2 \end{cases}$
2. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 - 3x + 1 \end{cases}$
3. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \sin(x) \end{cases}$
4. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \exp(x^2) \end{cases}$
5. $f : \begin{cases} \mathbb{R}_{>0} \to \mathbb{R} \\ x \mapsto \ln(x) \end{cases}$
6. $f : \begin{cases} [0, \frac{\pi}{4}] \to \mathbb{R} \\ x \mapsto \tan(x) \end{cases}$

Problem 11.5: Check whether the following functions are convex or concave and find the x- and y-coordinates of all inflection and saddle points.

1. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 \end{cases}$
2. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 - x + 1 \end{cases}$
3. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^4 + x^2 - 2 \end{cases}$
4. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 - 6x^2 + 12x - 7 \end{cases}$
5. $f : \begin{cases} \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R} \\ x \mapsto x - \tan(x) \end{cases}$

Problem 11.6: Analyse discontinuities.

1. $f : \begin{cases} \mathbb{R} \setminus \{-1\} \to \mathbb{R} \\ x \mapsto \dfrac{x-1}{x+1} \end{cases}$
2. $f : \begin{cases} \mathbb{R} \setminus \{-1\} \to \mathbb{R} \\ x \mapsto \dfrac{x^2-1}{x+1} \end{cases}$
3. $f : \begin{cases} \mathbb{R} \setminus \{0\} \to \mathbb{R} \\ x \mapsto \dfrac{e^x-1}{x} \end{cases}$
4. $f : \begin{cases} \mathbb{R} \setminus \{0\} \to \mathbb{R} \\ x \mapsto \dfrac{|x|}{2x} \end{cases}$

Problem 11.7: Find all asymptotes.

1. $f : \begin{cases} \mathbb{R} \setminus \{-1\} \to \mathbb{R} \\ x \mapsto \dfrac{2x-1}{x+1} \end{cases}$
2. $f : \begin{cases} \mathbb{R} \setminus \{-1\} \to \mathbb{R} \\ x \mapsto \dfrac{x^2-x+1}{x+1} \end{cases}$
3. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \sin(x) \end{cases}$
4. $f : \begin{cases} \mathbb{R} \setminus \{0\} \to \mathbb{R} \\ x \mapsto \dfrac{\sin(x)}{x} \end{cases}$

Problem 11.8: Analyse the following functions with respect to all properties listed in the introduction of this chapter including drawing a sketch.

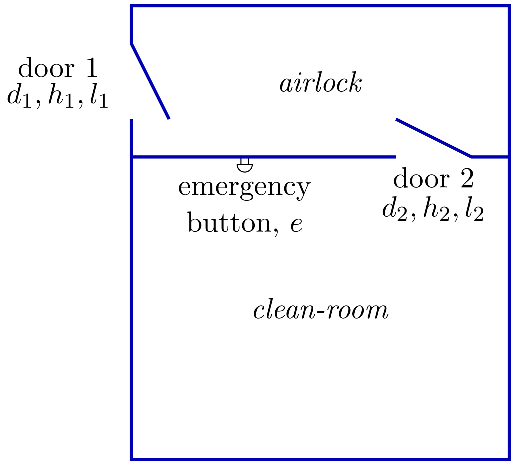
1. $f : \begin{cases} \mathbb{R} \setminus \{-1\} \to \mathbb{R} \\ x \mapsto \dfrac{1}{x+1} - 1 \end{cases}$
2. $f : \begin{cases} \mathbb{R} \setminus \{1\} \to \mathbb{R} \\ x \mapsto \dfrac{x^3-3x^2+4}{x^2-2x+1} \end{cases}$
3. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \sin(2x) + 1 \end{cases}$

12 Boolean algebra

12.1 Introduction

Example 12.1. You get the task to design a *clean-room* for industrial applications. You decide to have an *airlock* which is a small room between the clean-room and the outside world. The airlock has two doors and you have to make sure that only one door can be opened at once.

Each door is equipped with an electronic lock, a handle with a switch and another switch to indicate an open door. There exist also an emergency line which allows both doors to be opened at once.



An open door is indicated by $d_{1,2} = 1$, a pushed handle by $h_{1,2} = 1$, an emergency by $e = 1$ and the door is locket with $l_{1,2} = 1$. The opposite is marked with 0.

You find out that the lock of the first door l_1 depends on the handle of that door h_1, the state of the other door d_2 and the emergency line e. After thinking in detail about all possible options you derive the following table:

h_1	d_2	e	l_1
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

You recall *boolean algebra* an derive the conjunctive normal form for the lock of the first door:

$$l_1 = (\overline{h_1} \wedge \overline{d_2} \wedge \overline{e}) \vee (\overline{h_1} \wedge d_2 \wedge \overline{e}) \vee (h_1 \wedge d_2 \wedge \overline{e})$$

After playing around with some rules for Boolean functions you conclude for both doors with:

$$l_1 = (\overline{h_1} \vee d_2) \wedge \overline{e} \qquad l_2 = (\overline{h_2} \vee d_1) \wedge \overline{e}$$

◁

12.2 Boolean functions

Definition 12.1 (Boolean function)**.** Let $B = \{0,1\}$ be a set, then $f : B^n \to B$ is called a *Boolean function*. I.e. f assigns to every n-tupel $(x_1, x_2, \ldots, x_n)$ with $x_i \in B$ a value $f(x_1, x_2, \ldots, x_n) \in B$. ◁

Remark: Boolean algebra is not limited to the mentioned set, however, with digital technology in mind we limit ourself to this kind of sets.

Example 12.2. Let $f : B \to B, B = \{0,1\}$ be a Boolean function that maps every (single) element of B to another element of B. Since B has exactly two elements, there exist four possible functions $f(x)$, i.e. $f_0, \ldots, f_3$:

x	f_0	f_1	f_2	f_3
0	0	0	1	1
1	0	1	0	1

$f_0 = 0$ and $f_3 = 1$ are *constant functions*. $f_1 = x$ is the *identity function* and $f_2 = \overline{x}$ is the *negation function*. ◁

Example 12.3. Let $f : B \times B \to B, B = \{0,1\}$ be a Boolean function that maps any two elements of B to an element of B. Since B has exactly two elements, there exist sixteen possible functions $f(x)$, i.e. $f_0, \ldots, f_{16}$ as listed in table 12.1.

Some of the functions play an important role in Boolean algebra:

x_1	x_2	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Table 12.1: All possible Boolean functions $f : B^2 \to B$, $B = \{0, 1\}$.

$f(x)$	name	Notation
f_1	*AND (conjunction)*	$x_1 \wedge x_2$
f_6	*XOR (exclusive OR)*	$x_1 \not\leftrightarrow x_2$
f_7	*OR (disjunction)*	$x_1 \vee x_2$
f_8	*NOR (negated OR)*	$\overline{x_1 \vee x_2}$
f_9	*XNOR (negated XOR)*	$x \leftrightarrow x_2$
f_{10}, f_{12}	*NOT (negation)*	$\overline{x_2}, \overline{x_1}$
f_{14}	*NAND (negated AND)*	$\overline{x_1 \wedge x_2}$

◁

In chapter 1 of this lecture notes we learned some basic calculation rules for logic expressions. If we take 1 as a *true* statement and 0 as a *false* statement we may apply all these rules to Boolean algebra. Of particular interest are conjunction, disjunction and negation.

Theorem 12.2 (Laws on Boolean functions)**.** Let $x_1, x_2, x_3 \in \{0, 1\}$, then we have:

1. Associativity:

$$(x_1 \wedge x_2) \wedge x_3 = x_1 \wedge (x_2 \wedge x_3)$$
$$(x_1 \vee x_2) \vee x_3 = x_1 \vee (x_2 \vee x_3)$$

2. Commutativity:

$$x_1 \wedge x_2 = x_2 \wedge x_1$$
$$x_1 \vee x_2 = x_2 \vee x_1$$

3. Distributivity over $\wedge$ and $\vee$:

$$x_1 \vee (x_2 \wedge x_3) = (x_1 \vee x_2) \wedge (x_1 \vee x_3)$$
$$x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3)$$

4. Double negation:

$$\overline{\overline{x_1}} = x_1;$$

5. De Morgan's laws:

$$\overline{x_1 \wedge x_2} = \overline{x_1} \vee \overline{x_2}$$
$$\overline{x_1 \vee x_2} = \overline{x_1} \wedge \overline{x_2}$$

◁

By these basic laws it is possible to replace a Boolean function by other Boolean functions.

Example 12.4. We want to express the term $x_1 \wedge x_2 \wedge x_3$ by binary disjunction and negation only.

$$\begin{array}{ll}
x_1 \wedge x_2 \wedge x_3 & \mid \text{De Morgan} \\
= \overline{\overline{x_1} \vee \overline{x_2}} \wedge x_3 & \mid \text{De Morgan} \\
= \overline{\overline{\overline{\overline{x_1} \vee \overline{x_2}}} \vee \overline{x_3}} & \mid \text{remove double negation} \\
= \overline{\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}} & \mid \text{group in pairs} \\
= \overline{((\overline{x_1} \vee \overline{x_2}) \vee \overline{x_3})} &
\end{array}$$

◁

Example 12.5. It can be shown that any Boolean function can be replaced by multiples of $f(x_1, x_2) = \overline{x_1 \wedge x_2}$ (i.e. NAND). Some examples:

$$\overline{x} = \overline{x \wedge 1}$$
$$x_1 \wedge x_2 = \overline{\overline{x_1 \wedge x_2} \wedge 1}$$
$$x_1 \vee x_2 = \overline{\overline{x_1 \wedge 1} \wedge \overline{x_2 \wedge 1}}$$
$$x_1 \wedge x_2 \wedge x_3 = \overline{\overline{\overline{\overline{x_1 \wedge x_2} \wedge 1} \wedge x_3} \wedge 1}$$

◁

12.3 Normal form of functions

Is it possible to express any Boolean function in a unique normalized form? There are two ways to achieve this which are the so called *conjunctive-* and *disjunctive normal form.* To get there we first need to numerate all possible input combinations and to define the so called *min-* and *max-terms.*

Let us assume we have n input variables $x_0, x_1, \ldots, x_{n-1} \in \{0, 1\}$. For the numbering we arrange them in reversed order without commas $x_{n-1} \ldots x_1 x_0$ and treat them as a binary number. Listed in a table we get 2^n rows with row number 0 to $2^n - 1$.

row	x_2	x_1	x_0	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	M_0	M_1	M_2	M_3	M_4	M_5	M_6	M_7
0	0	0	0	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
1	0	0	1	0	1	0	0	0	0	0	0	1	0	1	1	1	1	1	1
2	0	1	0	0	0	1	0	0	0	0	0	1	1	0	1	1	1	1	1
3	0	1	1	0	0	0	1	0	0	0	0	1	1	1	0	1	1	1	1
4	1	0	0	0	0	0	0	1	0	0	0	1	1	1	1	0	1	1	1
5	1	0	1	0	0	0	0	0	1	0	0	1	1	1	1	1	0	1	1
6	1	1	0	0	0	0	0	0	0	1	0	1	1	1	1	1	1	0	1
7	1	1	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0

Table 12.2: Min- and max-terms for three Boolean variables $x_2, x_1, x_0 \in \{0, 1\}$.

row	$x_{n-1}x_{n-2}\dots x_1x_0$
0^{th}	$00\dots00$
1^{st}	$00\dots01$
2^{nd}	$00\dots10$
$\vdots$	$\vdots$
$2^n - 2^{\text{th}}$	$11\dots10$
$2^n - 1^{\text{th}}$	$11\dots11$

We get for the row number i:

$$i = \sum_{k=0}^{n-1} x_k 2^k$$

Definition 12.3 (Min- and max-term)**.** Let $B = \{0, 1\}$ be a Boolean set and $x_{n-1}, \dots, x_0 \in B$, $n \in \mathbb{N}$ be n Boolean variables.

We define a *min-term* m_i, $i = 0, 1, \dots, 2^n - 1$ with:

$$m_i := \begin{cases} B^n \to B & \\ x_{n-1}, \dots, x_0 \mapsto 1 & \text{if } i = \sum_{k=0}^{n-1} x_k 2^k \\ x_{n-1}, \dots, x_0 \mapsto 0 & \text{otherwise} \end{cases}$$

We define a *max-term* M_i, $i = 0, 1, \dots, 2^n - 1$ with:

$$M_i := \begin{cases} B^n \to B & \\ x_{n-1}, \dots, x_0 \mapsto 0 & \text{if } i = \sum_{k=0}^{n-1} x_k 2^k \\ x_{n-1}, \dots, x_0 \mapsto 1 & \text{otherwise} \end{cases}$$

◁

Example 12.6. For two Boolean variables x_1 and x_0 we have four min- and max-terms each as listed below.

row	x_1	x_0	m_0	m_1	m_2	m_3	M_0	M_1	M_2	M_3
0	0	0	1	0	0	0	0	1	1	1
1	0	1	0	1	0	0	1	0	1	1
2	1	0	0	0	1	0	1	1	0	1
3	1	1	0	0	0	1	1	1	1	0

◁

Example 12.7. For three Boolean variables x_2, x_1 and x_0 we have eight min- and eight max-terms, see table 12.2. ◁

Remark: Min- and max-terms can be expressed by the basic Boolean functions *negation*, *conjunction* and *disjunction*: We write the index i of the min- or max-term in binary form and assign the digits of the binary number to the Boolean variables $x_{n-1} \dots x_1 x_0$.

For min-terms we *negate* the variables with binary digit 0 and then take the *conjunction* of all variables.

For max-terms we *negate* the variables with binary digit 1 and then take the *disjunction* of all variables.

Example 12.8. For three variables x_2, x_1 and x_0 we get eight min-terms and eight max-terms. The expressions by negation, conjunction and disjunction are given by:

i	m_i	M_i
$0_{10} = 000_2$	$\overline{x_2} \wedge \overline{x_1} \wedge \overline{x_0}$	$x_2 \vee x_1 \vee x_0$
$1_{10} = 001_2$	$\overline{x_2} \wedge \overline{x_1} \wedge x_0$	$x_2 \vee x_1 \vee \overline{x_0}$
$2_{10} = 010_2$	$\overline{x_2} \wedge x_1 \wedge \overline{x_0}$	$x_2 \vee \overline{x_1} \vee x_0$
$3_{10} = 011_2$	$\overline{x_2} \wedge x_1 \wedge x_0$	$x_2 \vee \overline{x_1} \vee \overline{x_0}$
$4_{10} = 100_2$	$x_2 \wedge \overline{x_1} \wedge \overline{x_0}$	$\overline{x_2} \vee x_1 \vee x_0$
$5_{10} = 101_2$	$x_2 \wedge \overline{x_1} \wedge x_0$	$\overline{x_2} \vee x_1 \vee \overline{x_0}$
$6_{10} = 110_2$	$x_2 \wedge x_1 \wedge \overline{x_0}$	$\overline{x_2} \vee \overline{x_1} \vee x_0$
$7_{10} = 111_2$	$x_2 \wedge x_1 \wedge x_0$	$\overline{x_2} \vee \overline{x_1} \vee \overline{x_0}$

◁

Theorem 12.4 (Disjunctive normal form)**.** Any Boolean function can uniquely be expressed by a disjunction of min-terms. ◁

Example 12.9. We want to determine the disjunctive normal form for:

$$f : \begin{cases} B^3 \to B \\ x_2, x_1, x_0 \mapsto (x_2 \vee \overline{x_1}) \wedge x_0 \end{cases}$$

We derive the value table and add the minterms:

i	$x_2x_1x_0$	$f(x_2, x_1, x_0)$	m_1	m_5	m_7
0	0 0 0	0	0	0	0
1	0 0 1	1	1	0	0
2	0 1 0	0	0	0	0
3	0 1 1	0	0	0	0
4	1 0 0	0	0	0	0
5	1 0 1	1	0	1	0
6	1 1 0	0	0	0	0
7	1 1 1	1	0	0	1

Hence we get

$$\begin{aligned} f &= m_1 \vee m_5 \vee m_7 \\ &= (\overline{x_2} \wedge \overline{x_1} \wedge x_0) \vee (x_2 \wedge \overline{x_1} \wedge x_0) \\ &\quad \vee (x_2 \wedge x_1 \wedge x_0) \end{aligned}$$

which is the *disjunctive normal form* of f. ◁

Theorem 12.5 (Conjunctive normal form)**.** Any Boolean function can uniquely be expressed by a conjunction of max-terms. ◁

Example 12.10. We want to determine the conjunctive normal form for:

$$f : \begin{cases} B^3 \to B \\ x_2, x_1, x_0 \mapsto x_2 \vee (\overline{x_1} \wedge x_0) \end{cases}$$

We derive the value table with the corresponding max-terms:

i	$x_2x_1x_0$	$f(x_2, x_1, x_0)$	M_0	M_2	M_3
0	0 0 0	0	0	1	1
1	0 0 1	1	1	1	1
2	0 1 0	0	1	0	1
3	0 1 1	0	1	1	0
4	1 0 0	1	1	1	1
5	1 0 1	1	1	1	1
6	1 1 0	1	1	1	1
7	1 1 1	1	1	1	1

Hence we get

$$\begin{aligned} f &= M_0 \wedge M_2 \wedge M_3 \\ &= (x_2 \vee x_1 \vee x_0) \wedge (x_2 \vee \overline{x_1} \vee x_0) \\ &\quad \wedge (x_2 \vee \overline{x_1} \vee \overline{x_0}) \end{aligned}$$

which is the *conjunctive normal form* of f. ◁

12.4 Problems

Problem 12.1: Consider a Boolean function

$$f : \begin{cases} B^3 \to B, B = \{0, 1\} \\ (x_1, x_2, x_3) \mapsto f(x_1, x_2, x_2) \end{cases}$$

How many different input value combinations exist? How many different functions f are possible?

Problem 12.2: Which of the following equations are true? check by truth table.

1. $x \wedge y = \overline{\overline{x} \vee \overline{y}}$
2. $x \vee y = \overline{\overline{x} \wedge \overline{y}}$
3. $x \leftrightarrow y = (x \wedge y) \vee (\overline{x} \wedge \overline{y})$
4. $x \nleftrightarrow y = (x \vee y) \wedge (\overline{x} \vee \overline{y})$
5. $x \leftrightarrow y = (x \vee \overline{y}) \wedge (\overline{x} \vee y)$
6. $x \nleftrightarrow y = (x \wedge \overline{y}) \vee (\overline{x} \wedge y)$

Problem 12.3: Express the following terms by disjunction and negation only. (Take the outcome of the previous problem into account.)

1. $x_1 \wedge x_2$
2. $x_1 \leftrightarrow x_2$
3. $x_1 \wedge x_2 \wedge x_3$
4. $x_1 \nleftrightarrow x_2$

Problem 12.4: Express the following terms by binary negated conjunction (NAND) only: $f : B^2 \to B, (x, y) \mapsto \overline{x \wedge y}$.

1. $x_1 \vee x_2$
2. $\overline{x_1 \vee x_2}$
3. $x_1 \wedge \overline{x_2}$
4. $\overline{x_1} \vee \overline{x_2}$

Problem 12.5: Express the following terms in conjunctive normal form.

1. $x_1 \leftrightarrow x_2$
2. $x_0 \nleftrightarrow x_1$
3. $(x \vee y) \wedge z$
4. $(\overline{a} \wedge b) \vee \overline{c}$

Problem 12.6: Express the following terms in disjunctive normal form.

1. $x_1 \leftrightarrow x_2$
2. $x_0 \nleftrightarrow x_1$
3. $x \vee (y \wedge z)$
4. $\overline{a} \wedge (b \vee \overline{c})$

13 System of linear equations

13.1 Introduction

Example 13.1. “If John gave 1 € to Judy they both would have the same amount of money. If Judy gave 1 € to John he would have twice as much money as her. How much money does each of them have?”

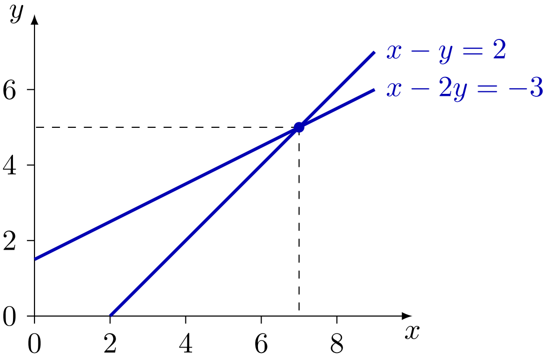
How do we find out? We first define x and y as John’s and Judy’s amount of money, respectively. Then we express the first two sentences as two equations:

$$\begin{aligned} x-1&=y+1\\ x+1&=2\cdot(y-1)\end{aligned}$$

We multiply the brackets, bring all constant summands to the right and all summands including x or y to the left:

$$\begin{aligned} x-y&=2\\ x-2y&=-3\end{aligned}$$

These two equations can be visualized as straight lines in a Cartesian diagram:



We subtract the first from the second equation and leave the first as it is:

$$\begin{aligned} x-y&=2\\ -y&=-5\end{aligned}$$

Then we subtract the second from the first equation and multiply the second by −1:

$$\begin{aligned} x&=7\\ y&=5\end{aligned}$$

Hence, John has 7 € and Judy has 5 €. ◁

This example deals with a small system of linear equations. With two equations and two unknowns we found a unique solution. Is this always possible? Is there always a unique solution?

A system of linear equations may deal with more than two unknowns. It may be used to numerically solve integrals or differential equations. In three dimensional field problems the number of unknowns may increase to several million or more!

In this chapter we get a first idea on systems of linear equations, derive a general approach to solve them and discuss whether they are solvable at all.

13.2 Definition

Definition 13.1 (System of linear equations)**.** Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. We define a *system of linear equations* (*SLE*) as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1\\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2\\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots\\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

with the coefficients $a_{11},a_{12},\dots,a_{mn}\in\mathbb{K}$, the constant terms $b_1,b_2,\dots,b_m\in\mathbb{K}$, $m\in\mathbb{N}$ the number of equations and $n\in\mathbb{N}$ the number of unknowns $x_1,\dots,x_n\in\mathbb{K}$.

If $b_1,\dots,b_m$ are all zero we call it a *homogeneous SLE* and a *inhomogeneous SLE* otherwise. ◁

Example 13.2. In the introductory example we derived an SLE with two equations containing the two unknowns x and y:

$$\begin{aligned} x-y&=2\\ x-2y&=-3\end{aligned}$$

Hence we get

$$\begin{aligned} a_{11}&=1, & a_{12}&=-1, & b_1&=2,\\ a_{21}&=1, & a_{22}&=-2, & b_2&=-3.\end{aligned}$$

◁

Remark: An SLE may be expressed by

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$A \qquad\qquad x \quad = \quad b$$

where A denotes the *coefficient matrix*, x a *solution vector* and b the *constant vector* or *inhomogeneous vectorinhomogeneous!vector*.

Example 13.3. We rewrite the SLE of the introductory example

$$\begin{aligned} x_1 - x_2 &= 2 \\ x_1 - 2x_2 &= -3 \end{aligned}$$

and get

$$\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$A \qquad\qquad x \quad = \quad b$$

◁

Remark: As an expression for an SLE we may write in short

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right) = (A \mid b)$$

which is the *extended coefficient matrix*.

Example 13.4. For the introductory example we write:

$$\left(\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & -2 & -3 \end{array}\right) = (A \mid b)$$

◁

13.3 Gauss-Jordan elimination

13.3.1 Elementary row operations

To solve an SLE we apply *elementary row operations*:

Theorem 13.2 (Row switching)**.** If an equation of an SLE is switched with another one, the values of the unknowns do not change. ◁

Theorem 13.3 (Row multiplication)**.** If an equation of an SLE is multiplied by a constant $c \in \mathbb{C}$, $c \neq 0$, then the values of the unknowns do not change. ◁

Theorem 13.4 (Row addition)**.** If an equation of an SLE is added to another equation of the same SLE, then the values of the unknowns do not change. ◁

13.3.2 An example

Let's assume the following SLE:

$$\begin{array}{rcrcrcr} 2x_1 & - & 2x_2 & + & x_3 & = & -3 \\ -x_1 & + & 3x_2 & + & x_3 & = & 3 \\ x_1 & + & 3x_2 & - & 2x_3 & = & 7 \end{array}$$

and write it as extended coefficient matrix:

$$\left(\begin{array}{rrr|r} 2 & -2 & 1 & -3 \\ -1 & 3 & 1 & 3 \\ 1 & 3 & -2 & 7 \end{array}\right)$$

First, where possible we change the elements at the bottom left to zeros. This is done by

1. multiplying a row by a factor to get a 1 for the first non-zero element and
2. subtracting rows to gain zero elements.

We divide row 1 by 2:

$$\left(\begin{array}{rrr|r} 1 & -1 & \frac{1}{2} & -\frac{3}{2} \\ -1 & 3 & 1 & 3 \\ 1 & 3 & -2 & 7 \end{array}\right)$$

Add row 1 to row 2 and subtract row 1 from row 3:

$$\left(\begin{array}{rrr|r} 1 & -1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 2 & \frac{3}{2} & \frac{3}{2} \\ 0 & 4 & -\frac{5}{2} & \frac{17}{2} \end{array}\right)$$

Divide row 2 by 2:

$$\left(\begin{array}{rrr|r} 1 & -1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & \frac{3}{4} & \frac{3}{4} \\ 0 & 4 & -\frac{5}{2} & \frac{17}{2} \end{array}\right)$$

Subtract the quadruple of row 2 from row 3:

$$\left(\begin{array}{rrr|r} 1 & -1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & \frac{3}{4} & \frac{3}{4} \\ 0 & 0 & -\frac{11}{2} & \frac{11}{2} \end{array}\right)$$

Divide row 3 by $-\frac{11}{2}$:

$$\left(\begin{array}{ccc|c} 1 & -1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & \frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Now we concentrate on the elements at the top right. We subtract $-\frac{1}{2}$ of row 3 from row 1 and subtract $\frac{3}{4}$ of row 3 from row 2:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Add row 2 to row 1:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -1 \end{array}\right)$$

With this extended coefficient matrix we get the solution:

$$x_1 = \frac{1}{2} \qquad x_2 = \frac{3}{2} \qquad x_3 = -1$$

In this example we were lucky to end with an identity matrix, i.e.

$$a_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{otherwise} \end{cases}$$

This is not always the case. To find a general approach we need the so called *reduced row echelon form* of a matrix:

13.3.3 Reduced row echelon form, rref

Definition 13.5 (Reduced row echelon form)**.** Let $M(m \times n, \mathbb{K})$ be a matrix with m rows and n columns of elements in $\mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We say M has *reduced row echelon form* (*rref*) if

1. All rows containing zeros only (i.e. zero-rows) are at the end of M (i.e. the bottom of M).
2. The leading entry of non-zero rows (i.e. the first from the left unequal zero) is one.
3. From row 2 on the leading entry of non-zero rows is further to the right than all previous leading entries (i.e. the rows above).
4. All entities above and below a leading one are zero.

◁

Remark: In case of an extended coefficient matrix $(A \mid b)$ the definition applies to the coefficients a_{ij} only and not to the constant terms b_i.

Theorem 13.6 (Reduced row echelon form)**.** Any matrix $M(m \times n, \mathbb{K})$ can be changed into reduced row echelon form by elementary row operations. ◁

To change a matrix into the reduced row echelon form we proceed as in the previous example. First we change the elements at bottom left to zeros:

1. If there exists a row below the first with a leading entry further to the left than in the first row switch the rows.
2. Divide the first row by its leading entry to change it to unity.
3. Subtract multiples of row 1 from the subsequent rows to change all elements below the leading entry of row 1 to zero.
4. Repeat this three steps for rows 2, 3, 4 etc.

Then we convert the elements above the leading entries to zeros:

1. Subtract multiples of the last non-zero row, say row k, from the rows above to change the elements above the leading entry to zero.
2. Repeat this step for all rows above row k.

Remark: To indicate the change of a matrix A into the reduced row echelon form we may write in short:

$$\text{rref}(A)$$

Example 13.5. We want to change the SLE

$$\begin{array}{ccccccccc} & & 2x_2 & + & 3x_3 & + & 4x_4 & = & 1 \\ x_1 & + & 4x_2 & + & 3x_3 & + & 5x_4 & = & 6 \\ x_1 & + & 4x_2 & + & 5x_3 & + & 1x_4 & = & 4 \\ x_1 & + & 3x_2 & + & 2x_3 & + & 2x_4 & = & 5 \end{array}$$

into the reduced row echelon form and write it as extended coefficient matrix:

$$\left(\begin{array}{cccc|c} 0 & 2 & 3 & 4 & 1 \\ 1 & 4 & 3 & 5 & 6 \\ 1 & 4 & 5 & 1 & 4 \\ 1 & 3 & 2 & 2 & 5 \end{array}\right)$$

We proceed as follows: switch rows 1 and 2:

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 5 & 6 \\ 0 & 2 & 3 & 4 & 1 \\ 1 & 4 & 5 & 1 & 4 \\ 1 & 3 & 2 & 2 & 5 \end{array}\right)$$

Subtract row 1 from row 3 and 4:

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 5 & 6 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & 0 & 2 & -4 & -2 \\ 0 & -1 & -1 & -3 & -1 \end{array}\right)$$

Divide row 2 by 2:

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 5 & 6 \\ 0 & 1 & \frac{3}{2} & 2 & \frac{1}{2} \\ 0 & 0 & 2 & -4 & -2 \\ 0 & -1 & -1 & -3 & -1 \end{array}\right)$$

Add row 2 to row 4:

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 5 & 6 \\ 0 & 1 & \frac{3}{2} & 2 & \frac{1}{2} \\ 0 & 0 & 2 & -4 & -2 \\ 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} \end{array}\right)$$

Divide row 3 by 2:

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 5 & 6 \\ 0 & 1 & \frac{3}{2} & 2 & \frac{1}{2} \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} \end{array}\right)$$

Subtract $\frac{1}{2}$ of row 3 from row 4:

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 5 & 6 \\ 0 & 1 & \frac{3}{2} & 2 & \frac{1}{2} \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Subtract $\frac{3}{2}$ of row 3 from row 2 and subtract row 3 three times from row 1:

$$\left(\begin{array}{cccc|c} 1 & 4 & 0 & 11 & 9 \\ 0 & 1 & 0 & 5 & 2 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Subtract row 2 four times from row 1:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -9 & 1 \\ 0 & 1 & 0 & 5 & 2 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This is the reduced row echelon form. ◁

13.4 Solutions of an SLE

13.4.1 Three solution behaviours

When trying to find the solution of an SLE we may find

1. no solution,
2. a single solution or
3. an infinite number of solutions.

For two unknowns this may be visualised in a diagram where each equation is represented by a straight line. The solutions are where the lines coincide.

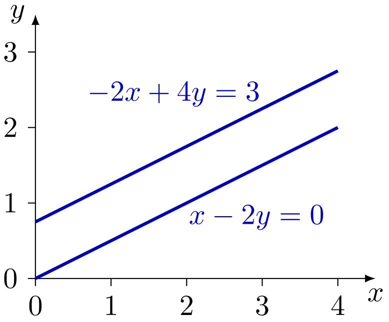
If the lines are parallel but not identical, there is no solution. If the lines intersect, there is a single solution. If the lines coincide, i.e. they have the same slope and position, there exist an infinite number of solutions along the line.

Example 13.6. We look at three examples with two equations and unknowns and visualize the equations as graphs.

- The system of linear equations

  $$x - 2y = 0 \qquad -2x + 4y = 3$$

  has no solution.

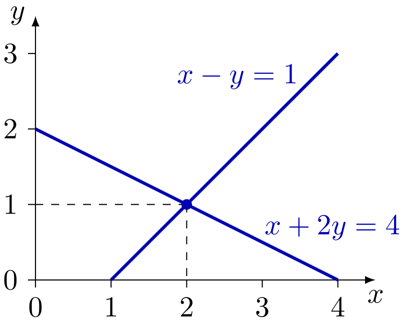


- The system of linear equations

  $$x - y = 1 \qquad x + 2y = 4$$

  has the single solution

  $$x = 2 \qquad y = 1$$

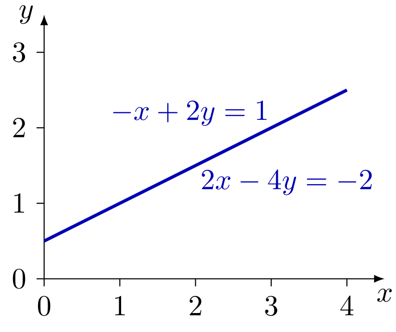


- The system of linear equations

  $$-x + 2y = 1 \qquad 2x - 4y = -2$$

  has an infinite number of solutions which may be expressed as:

  $$y = \frac{x+1}{2}$$

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Remark: For an infinite number of solutions one (or more) of the unknowns are used as an parameter to express the other unknowns.

13.4.2 Solution of a homogeneous SLE

The constant terms of a homogeneous SLE remain zero when performing elementary row operations. Hence, the Gauss-Jordan elimination may be performed on the coefficient matrix.

A homogeneous SLE has always a trivial solution: All unknowns are zero. Hence, a homogeneous SLE shows only two solution behaviours: a single solution or an infinite number of solutions. Also, if a homogeneous SLE has a single unique solution, then all unknowns are zero.

If the reduced row echelon form is the identity matrix:

$$\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

or an identity matrix with some zero-rows underneath:

$$\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

then there exist only the single solution where all unknowns are zero.

If there exist one ore more rows in the reduced row echelon form with more than one non-zero element, then the homogeneous SLE has an infinite number of solutions.

13.4.3 Solution of a inhomogeneous SLE

We now investigate the influence of the solution behaviour of an inhomogeneous SLE on the reduced row echelon form of the extended coefficient matrix $\mathrm{rref}(A \mid b)$.

If the reduced row echelon form of the extended coefficient matrix has one or more rows where only the constant is non-zero

$$0 \cdot x_1 + 0 \cdot x_2 + \ldots + 0 \cdot x_n \neq 0$$

we have a contradiction. Hence, the SLE has no solution.

If the reduced row echelon form is the identity matrix:

$$\mathrm{rref}(A \mid b) = \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_n \end{array}\right)$$

or the identity matrix with some zero-rows underneath:

$$\mathrm{rref}(A) = \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{array}\right)$$

then, there exist a single solution.

If the reduced row echelon form does not contradict itself, and there are one or more rows with more than one non-zero coefficients, then there are an infinite number of solutions.

Example 13.7. From the previous example we get the reduced row echelon form of the extended coefficient matices:

- $\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 3 \end{array}\right)$

  The last row leads to a contradiction and, hence, the SLE has no solution.

- $\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$

  We see the identity matrix for the coefficients and, hence, the SLE has a single solution.
- $\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right)$

  There is no contradiction and row 1 has two non-zero coefficients, hence, the SLE has an infinite number of solutions.

◁

13.5 Problems

Problem 13.1: Solve the following SLE by Gauss-Jordan-elimination using the extended coefficient matrix.

1. $$\begin{aligned} 2x + y + z &= 7 \\ x + 2y + z &= 8 \\ x + y + 2z &= 9 \end{aligned}$$
2. $$\begin{aligned} a - b + c &= 1 \\ a + b - c &= 1 \\ a + b + c &= -1 \end{aligned}$$
3. $$\begin{aligned} 5x_1 + 4x_2 + 3x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= 4 \\ x_1 - x_2 - 2x_3 &= 6 \end{aligned}$$
4. $$\begin{aligned} 2a + 4b - c &= 2 \\ 5a - 3b + 2d &= 3 \\ 4a - 2c + d &= -1 \\ b + 2c - d &= 2 \end{aligned}$$
5. $$\begin{aligned} 4x_1 + 3x_2 + 2x_3 + x_4 &= -2 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 2 \\ x_1 + 2x_2 + x_3 + 2x_4 &= 2 \\ x_1 + x_2 + 2x_3 + 3x_4 &= 1 \end{aligned}$$
6. $$\begin{aligned} 2u + v + w + x &= 6 \\ u - v + w - 3x &= 5 \\ 2u + v - w - 2x &= 3 \\ u + 2v - 2w - x &= 0 \end{aligned}$$

Problem 13.2: Solve the following SLE by changing them into reduced row echelon form. Explain the results.

1. $$\begin{aligned} x + y &= 1 \\ y + z &= -1 \\ x - z &= 1 \end{aligned}$$
2. $$\begin{aligned} a + b &= 1 \\ b + c &= 1 \\ a + c &= 0 \end{aligned}$$
3. $$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= 8 \\ x_1 + x_2 + x_3 &= 3 \end{aligned}$$
4. $$\begin{aligned} a + 2b + 3c + 4d &= -7 \\ 3a + 2b + c + d &= 5 \\ 3a + 2b + c + 2d &= 3 \\ a + b + c + d &= 0 \end{aligned}$$
5. $$\begin{aligned} 2x_1 + x_2 - x_3 - 2x_4 &= -1 \\ 3x_1 + x_2 - x_3 - 3x_4 &= -2 \\ x_1 - x_2 - x_3 + x_4 &= 0 \\ x_1 + x_2 - x_3 - x_4 &= 1 \end{aligned}$$

14 Matrices

14.1 Introduction

In the previous chapter we already expressed a system of linear equations SLE by a matrix and two column-vectors $A \cdot x = b$. However, we used it only as a technique to write our SLE without extra addition and equality signs $+$ and $=$.

In this chapter we will see the meaning behind this notation an learn basic operations on matrices. We will see that the inverse matrix A^{-1} is a useful tool to evaluate the unknowns for many different constants b. And finally, we will look again on the solution behaviour of SLEs by analysing its (extended) coefficient matrix.

14.2 Definition and special matrices

Definition 14.1 ($m \times n$-matrix)**.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A rectangular arrangement of numbers $a_{ij} \in \mathbb{K},\ \ i = 1, \dots, n,\ \ j = 1, \dots, m,\ \ m, n \in \mathbb{N}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called a *$m \times n$-matrix*. We call a_{ij} the *elements* of A, m the number of rows and n the number of columns. The set of all possible $m \times n$-matrices is denoted by $M(m \times n, \mathbb{K})$. In short we write

$$A = (a_{ij})_{m,n} \qquad \text{or just} \qquad A = (a_{ij})$$

We call $r_i := (a_{i1}, a_{i2}, \dots, a_{in})$ the i^{th} *row-vector* of A and write

$$A = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{pmatrix}$$

We call

$$c_j := \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} = (a_{1j}, a_{2j}, \dots, a_{mj})^T$$

the j^{th} *column-vector* of A and write:

$$A = ((c_1, c_2, \dots, c_n))$$

The double brackets indicate a matrix in contrast to a vector. ◁

Definition 14.2 (Special matrices)**.** Some special matrices:

- All elements of the *zero-matrix* $0_{m,n} \in M(m \times n, \mathbb{K})$ are zero, e.g.

  $$0_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad 0_{3,4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- For a *square-matrix* the number of rows and columns are equal, i.e. $M(n \times n, \mathbb{K})$. E.g.

  $$A = (a_{ij})_{3,3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

  In short we may write: $A = (a_{ij})_3$

- A *diagonal matrix* is a (usually square) matrix $A = (a_{ij})_{m,n}$ with

  $$a_{ij} = \begin{cases} * & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

  where $*$ denotes any value (including zero). E.g.:

  $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

- The *identity matrix* $I_n \in M(n \times n, \mathbb{K})$ or $I_{m,n} \in M(m \times n, \mathbb{K})$ is a diagonal matrix with the diagonal elements being one:

  $$a_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

  E.g.:

  $$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad I_{3,2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

We call the column-vectors of the identity matrix *unit-vector*. I.e. the indintity matrix I_n has the unit vectors $e_1, e_2, \ldots, e_n$:

$$I_n = ((e_1, e_2, \ldots, e_n))$$

- A *lower triangular matrix* or *left triangular matrix* denotes a matrix $A = (a_{ij})$ where all elements a_{ij} with $i < j$ are zero, i.e.

  $$a_{ij} = \begin{cases} 0 & \text{for } i < j \\ * & \text{for } i \geq j \end{cases}$$

  where $*$ denotes any value (including zero). E.g.

  $$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 2 & 5 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \end{pmatrix}$$

- An *upper triangular matrix* or *right triangular matrix* denotes a matrix $A = (a_{ij})$ where all elements a_{ij} with $i > j$ are zero, i.e.

  $$a_{ij} = \begin{cases} 0 & \text{for } i > j \\ * & \text{for } i \leq j \end{cases}$$

  where $*$ denotes any value (including zero). E.g.

  $$\begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 0 & 4 & 1 \end{pmatrix}$$

◁

14.3 Basic matrix operations

Definition 14.3 (Sum of matrices)**.** Two matrices $A = (a_{ij})_{m,n}$ and $B = (b_{ij})_{m,n}$ are added by adding their elements:

$$A + B = (a_{ij} + b_{ij})_{m,n}$$

◁

Example 14.1.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 6 \\ 3 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ 6 & 7 \\ 5 & 7 \end{pmatrix}$$

◁

Definition 14.4 (Multiplication with scalar)**.** A matrix $A = (a_{ij})_{m,n}$ is multiplied by a scalar $\lambda \in \mathbb{K}$ by multiplying its elements with λ:

$$\lambda \cdot A = (\lambda \cdot a_{ij})_{m,n}$$

◁

Example 14.2.

$$3 \cdot \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 3 \\ 9 & 6 & 3 \end{pmatrix}$$

◁

Definition 14.5 (Transpose of a matrix)**.** The *transpose* of a matrix $A = (a_{ij})_{m,n}$ is a $n \times m$-matrix with switched rows and columns, i.e.

$$A^T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

◁

Example 14.3. With

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

we get

$$A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

◁

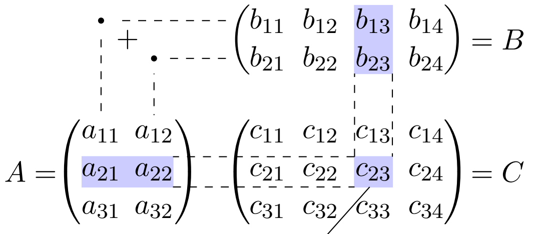
14.4 Matrix multiplication

Definition 14.6 (Matrix product)**.** Let $A = (a_{ij})_{m,n}$ and $B = (b_{ij})_{n,p}$ be two matrices. We define the product $C = (c_{ij})_{m,p}$ of A and B by

$$C = A \cdot B$$
$$(c_{ij})_{m,p} = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)_{m,p}$$

◁

Remark: The number of columns of the left matrix must equal the number of rows of the right matrix. The product may be written without the dot: $A \cdot B = AB$.



$$c_{23} = a_{21} \cdot b_{13} + a_{22} \cdot b_{23}$$

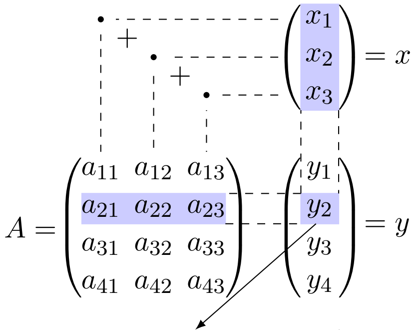
Theorem 14.7 (Calculation rules for matrix product)**.** For the matrix product we have:

- With $A, B \in M(m \times n, \mathbb{K})$ and $C, D \in M(n \times p, \mathbb{K})$ we have:
  $$A(C + D) = AC + AD \qquad \text{and}$$
  $$(A + B)C = AC + BC$$
- With $A \in M(m \times n, \mathbb{K})$ and $B \in M(n \times p, \mathbb{K})$ and $\lambda \in \mathbb{K}$ we have:
  $$A(\lambda B) = (\lambda A)B = \lambda(AB)$$
- With $A \in M(m \times n, \mathbb{K})$, $B \in M(n \times p, \mathbb{K})$ and $C \in M(p \times q, \mathbb{K})$ we have:
  $$(AB)C = A(BC)$$
- With $A \in M(m \times n, \mathbb{K})$ we have:
  $$A \cdot I_n = A \qquad \text{and} \qquad I_m \cdot A = A$$

◁

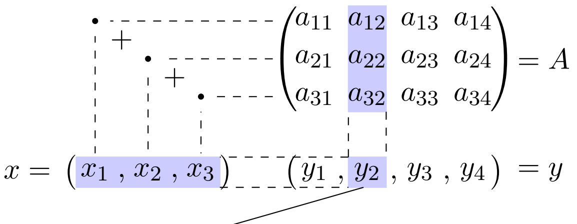
Column- or row-vectors may be looked at as matrices with one column or one row, respectively. Hence, we may multiply matrices with vectors under the same conditions as the matrix product.

The product of a matrix with a column-vector results in another column-vector. The number of columns of the matrix must equal the number of elements of the column-vector to be multiplied. The product column-vector has as many elements as the matrix has rows.



$$y_2 = a_{21} \cdot x_1 + a_{22} \cdot x_2 + a_{23} \cdot x_3$$

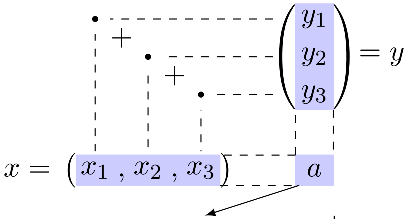
The product of a row-vector with a matrix results in another row-vector. The number of elements of the row-vector to be multiplied must equal the number of rows of the matrix. The product row-vector has as many elements as the matrix has columns.



$$y_2 = x_1 \cdot a_{12} + x_2 \cdot a_{22} + x_3 \cdot a_{32}$$

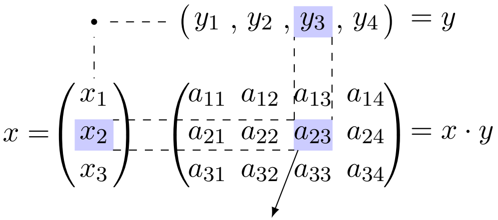
We may also multiply a row-vector with a column-vector and vice versa. Again, the number of columns of the left factor must equal the number of rows of the right factor.

If we multiply a row-vector with a column-vector, both with the same number of elements, the product is a scalar, i.e. an element of $\mathbb{K}$.



$$a = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$$

Multiplying a column-vector with a row-vector, both with arbitrary number of elements, results in a matrix. The number of rows of the matrix equals the number of elements of the column-vector. The number of columns of the matrix equals the number of elements of the row-vector.



$$a_{23} = x_2 \cdot y_3$$

Definition 14.8 (Inverse matrix)**.**

If for a square matrix $A \in (n \times n, \mathbb{K})$ there exist a matrix $B \in (n \times n, \mathbb{K})$ such that

$$A \cdot B = I_n$$

we say B is the *inverse* of A and denote the *inverse matrix* by A^{-1}. If for a matrix A there exist an inverse matrix A^{-1} we call it an *invertible matrix*. ◁

Theorem 14.9 (Calculation rules for inverse matrices)**.** For matrices $A, B \in M(n \times n, \mathbb{K})$ with existing inverse A^{-1}, B^{-1} we have:

- The inverse of the inverse is the matrix itself:
$$\left(A^{-1}\right)^{-1} = A$$
- The product of a matrix with its inverse is commutative:
$$A \cdot A^{-1} = A^{-1} \cdot A = I_n$$
- The inverse of a product of matrices is the product of the inverse matrices with switched factors:
$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$
- The identity matrix is the inverse of itself:
$${I_n}^{-1} = I_n$$

◁

Remark: We will see further down how to evaluate the inverse of a matrix.

14.5 System of linear equations and matrices

Example 14.4. Let us look at the following equation:

$$\begin{array}{ccccc} A & \cdot & x & = & b \\ \begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} & \cdot & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & = & \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \end{array}$$

We have a matrix $A = (a_{ij})_{4,3}$, a column-vector $x = (x_1, x_2, x_2)^T$ and another column-vector $b = (b_1, b_2, b_3, b_4)^T$. We solve this separately for the elements of b and get

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & b_3 \\ a_{41}x_1 & + & a_{42}x_2 & + & a_{43}x_3 & = & b_4 \end{array}$$

which is the set-up of a system on linear equations, SLE, as we introduced it in the last chapter. ◁

The example shows that we may use a matrix product to evaluate for a given matrix A and a column-vector x the column-vector b. However, for an SLE we want to do it the other way round: For a given matrix A and a column-vector b we want to derive the column-vector x. We take the matrix equation for the SLE and multiply both sides by the inverse matrix A^{-1}:

$$\begin{aligned} A \cdot x &= b \\ A^{-1} \cdot A \cdot x &= A^{-1} \cdot b \\ I_n \cdot x &= A^{-1} \cdot b \\ x &= A^{-1} \cdot b \end{aligned}$$

Hence, if we find the inverse A^{-1} of the coefficient matrix A of an SLE, we directly may derive the unknowns x from the constants b.

14.6 Matrix inversion

For an invertible square matrix $A = (a_{ij})_n$ we want to evaluate the inverse $A^{-1} = (b_{ij})_n$. By definition we have:

$$A \cdot A^{-1} = I_n$$

We express A^{-1} by its column-vectors $c_j = (b_{1j}, b_{2j}, \ldots, b_{nj})^T$:

$$\begin{aligned} A^{-1} &= \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \\ &= \left(\left(\begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix}, \ldots, \begin{pmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \end{pmatrix}\right)\right) \\ &= ((c_1, c_2, \ldots, c_n)) \end{aligned}$$

We do the same for the identity matrix I_n and express it by its unit-vectors e_j:

$$\begin{aligned} I_n &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \\ &= \left(\left(\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\right)\right) \\ &= ((e_1, e_2, \ldots, e_n)) \end{aligned}$$

Now we get

$$A \cdot ((c_1, c_2, \ldots, c_n)) = ((e_1, e_2, \ldots, e_n))$$

Column j of a matrix product depends only on the left factor and column j of the right factor. All other columns of the right factor have no influence on column j of the product. Hence, we may separate the equation into a set of n products of a matrix with a column-vector:

$$\begin{array}{ccccc} A & \cdot & c_j & = & e_j \\ A & \cdot & \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} & = & \begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow j^{\text{th}} \text{ row} \end{array}$$

for $j = 1, \ldots, n$.

Each of these n products may be solved for the column vector c_j like an SLE. However, since the process is quite similar each time we combine all SLEs into one large extended coefficient matrix:

$$\left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \ldots & a_{1n} & 1 & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & 0 & 0 & \ldots & 1 \end{array}\right)$$

After performing the Gauss-Jordan elimination we get on the right-hand-side the inverse matrix A^{-1}.

Example 14.5. For the SLE

$$\begin{array}{rcrcrcl} 2x & + & y & + & z & = & 7 \\ x & + & 2y & + & z & = & 8 \\ x & + & y & + & 2z & = & 9 \end{array}$$

we want to derive the inverse coefficient matrix. I.e. we neglect the constants in the first place and use them only at the end together with the inverse matrix to evaluate the unknowns.

We create the extended coefficient matrix:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

divide row 1 by 2:

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

subtract row 1 from row 2 and row 3:

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right)$$

multiply row 2 by $\frac{2}{3}$:

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right)$$

subtract $\frac{1}{2}$ of row 2 from row 3:

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right)$$

multiply row 3 by $\frac{3}{4}$:

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right)$$

subtract $\frac{1}{3}$ of row 3 from row 2 and $\frac{1}{2}$ of row 3 from row 1:

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{5}{8} & \frac{1}{8} & -\frac{3}{8} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right)$$

and subtract $\frac{1}{2}$ of row 2 from row 1:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right)$$

Thus, we get the inverse matrix:

$$A^{-1} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

With the inverse matrix A^{-1} and the constants $b = (7, 8, 9)^T$ we evaluate the three unknowns:

$$\begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

I.e.

$$\begin{aligned} x_1 &= \tfrac{3}{4} \cdot 7 - \tfrac{1}{4} \cdot 8 - \tfrac{1}{4} \cdot 9 = 1 \\ x_2 &= -\tfrac{1}{4} \cdot 7 + \tfrac{3}{4} \cdot 8 - \tfrac{1}{4} \cdot 9 = 2 \\ x_3 &= -\tfrac{1}{4} \cdot 7 - \tfrac{1}{4} \cdot 8 + \tfrac{3}{4} \cdot 9 = 3 \end{aligned}$$

$\triangleleft$

Example 14.6. The SLE

$$\begin{array}{rcrcrcl} 2x & + & y & + & z & = & 4 \\ x & + & 2y & + & z & = & 3 \\ x & + & y & + & 2z & = & 1 \end{array}$$

is equal to the SLE of the previous example except for the constant vector b. Hence, to derive the unknowns we may take the inverse matrix A^{-1} of the previous example and the constant vector $b = (4, 3, 1)^T$:

$$\begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

i.e. $x = 2$, $y = 1$ and $z = -1$. ◁

14.7 Rank of a matrix

If a row-vector of a matrix may be expressed by a sum of multiples of other row-vectors, we say the row is *linear dependent* on the other rows. In terms of solving an SLE such a row-vector does not contain any information that is not contained in the other row-vectors. We come to a general definition of linear dependence:

Definition 14.10 (Linear dependence and independence)**.** If for a set of vectors $v_1, \ldots, v_m \in \mathbb{K}^n$ there exist a set of scalars $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$ with at least one of them not being zero where

$$\lambda_1 \cdot v_1 + \lambda_2 \cdot v_2 + \ldots + \lambda_m \cdot v_m = 0$$

we say these vectors are *linear dependent.* If the above equation holds only for the trivial case, i.e. all scalars are zero, we say the vectors are *linear independent.* ◁

When changing into its reduced row echelon form, a matrix with linear dependent rows will show zero rows whereas a matrix with linear independent rows will show no zero rows.

Hence, the reduced row echelon form of a matrix shows by its zero rows whether rows of the matrix are linear dependent or not.

Linear dependent rows of an SLE may be removed without changing the values of its unknowns.

Theorem 14.11 (Number or linear independent rows and columns)**.** For a matrix $A \in M(m \times n, \mathbb{K})$ the number of linear independent rows and columns are the same. ◁

Definition 14.12 (Rank of a matrix)**.** We call the number of linear independent rows (or columns) the *rank of a matrix.*

For a matrix A we write in short: $\mathrm{rank}(A)$ ◁

Remark: The rank of a matrix equals the number of non-zero rows in its reduced row echelon form.

Theorem 14.13 (Rank and solutions of an SLE)**.** Let $A \in M(m \times n, \mathbb{K})$ be the coefficient matrix and $(A \mid b) \in M(m \times n + 1, \mathbb{K})$ be the extended coefficient matrix of an SLE with m equations and n unknowns. The SLE has

- no solution if
$$\mathrm{rank}(A) \neq \mathrm{rank}(A \mid b)$$
- a single solution if
$$\mathrm{rank}(A) = \mathrm{rank}(A \mid b) = n$$
- an infinite number of solutions if
$$\mathrm{rank}(A) = \mathrm{rank}(A \mid b) < n$$

◁

Example 14.7. We look at the following SLE:

$$\begin{array}{rcrcr} x & + & y & = & 1 \\ y & + & z & = & -1 \\ x & - & z & = & 1 \end{array}$$

and change its extended coefficient matrix into its reduced row echelon form:

$$\left(\begin{array}{rrr|r} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 1 \end{array}\right)$$

subtract row 1 from row 3:

$$\left(\begin{array}{rrr|r} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 0 \end{array}\right)$$

add row 2 to row 3:

$$\left(\begin{array}{rrr|r} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

subtract row 2 from row 1:

$$\left(\begin{array}{rrr|r} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

With A as the coefficient matrix and $(A \mid b)$ as the extended coefficient matrix of the SLE we get:

$$\operatorname{rank}(A) = 2 \qquad \operatorname{rank}(A \mid b) = 3$$

Hence, the SLE has no solution. ◁

Example 14.8. We look at the following SLE:

$$\begin{aligned} a + 2b + 3c &= 0 \\ 3a + 2b + c &= 4 \\ a + b + c &= 1 \end{aligned}$$

Changing its extended coefficient matrix into reduced row echelon form we get:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

With A as the coefficient matrix and $(A \mid b)$ as the extended coefficient matrix of the SLE we get:

$$\operatorname{rank}(A) = 2 \qquad \operatorname{rank}(A \mid b) = 2$$

Since the rank of A and $(A \mid b)$ are equal and less than the number of unknowns we get an infinite number of solutions. ◁

14.8 Problems

Problem 14.1: Which of the following statements are true?

1. A diagonal matrix is an identity matrix.
2. The transpose of a square matrix is a square matrix.
3. A diagonal matrix is a triangular matrix.
4. A matrix which is upper and lower triangular is an identity matrix.
5. An identity matrix is a diagonal matrix.
6. A left triangular matrix is an upper triangular matrix.
7. The transpose of an upper triangular matrix is a left triangular matrix.
8. A matrix which is upper and lower triangular is a diagonal matrix.
9. A zero matrix is a triangular matrix.
10. The transpose of an identity matrix is a diagonal matrix.

Problem 14.2: Let $A \in M(2 \times 3, \mathbb{K})$ and $B, C \in M(3 \times 4, \mathbb{K})$. Which of the following statements are true?

1. $A \cdot B \in M(2 \times 4, \mathbb{K})$
2. $B \cdot C \in M(3 \times 4, \mathbb{K})$
3. $A + B \in M(2 \times 4, \mathbb{K})$
4. $B + C \in M(3 \times 4, \mathbb{K})$
5. $A \cdot (B + C) \in M(2 \times 4, \mathbb{K})$
6. $B \cdot C^T \in M(3 \times 3, \mathbb{K})$
7. $B \cdot C^T \in M(4 \times 4, \mathbb{K})$
8. $B^T \cdot C \in M(3 \times 3, \mathbb{K})$
9. $B^T \cdot C \in M(4 \times 4, \mathbb{K})$

Problem 14.3: With

$$x = (1, 2, 3) \qquad y = (3, 3, 1)^T$$

perform the following multiplications:

1. $x \cdot y$
2. $y \cdot x$
3. $x^T \cdot y^T$
4. $y^T \cdot x^T$

Problem 14.4: With

$$\begin{aligned} x &= (1, 2, 3) \\ y &= (3, 4, 1, 2)^T \\ A &= \begin{pmatrix} 2 & 1 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \end{aligned}$$

perform the following multiplications:

1. A^T
2. $x \cdot A$
3. $A \cdot y$
4. $A \cdot I_4$

Problem 14.5: Evaluate the following products:

1. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 \\ 5 & 4 \\ 1 & 3 \end{pmatrix}$

3. $\begin{pmatrix} 2 & 4 \\ 5 & 4 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}$

Problem 14.6: Evaluate the inverse of the following matrices.

1. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

2. $\begin{pmatrix} 1/2 & 3/2 \\ 1/4 & 1/2 \end{pmatrix}$

3. $\begin{pmatrix} 1/2 & 1 & 1/2 \\ 1 & 1/2 & 3/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 3 & 2 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$

Problem 14.7: Evaluate the inverse of the following matrices.

1. $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 2 & 1 & 0 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 2 & -1 & -2 \\ -2 & 1 & -1 & 2 \\ -1 & -2 & 2 & 1 \\ 2 & -1 & 1 & -1 \end{pmatrix}$

3. $\frac{1}{40} \cdot \begin{pmatrix} -9 & 1 & 1 & 11 \\ 1 & 1 & 11 & -9 \\ 1 & 11 & -9 & 1 \\ 11 & -9 & 1 & 1 \end{pmatrix}$

4. $\begin{pmatrix} 1 & -1 & -1 & -2 \\ -1 & 2 & -1 & 1 \\ 1 & -1 & 2 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

Problem 14.8: Is it possible that the rank of the coefficient matrix of an SLE is larger than the number of unknowns?

Problem 14.9: Is it possible that the rank of the extended coefficient matrix of an SLE is larger than the number of unknowns?

Problem 14.10: Evaluate for the following SLEs the rank of the coefficient matrix and the rank of the extended coefficient matrix. What is the consequence for the solution of the SLEs?

1.
$$\begin{aligned} 2x - 2y + 2z &= 0 \\ x + 2y + 2z &= 1 \\ 3y + z &= 1 \end{aligned}$$

2.
$$\begin{aligned} 2a - 2b + 2c &= 6 \\ a + 2b + 2c &= 5 \\ 2a + b + 3c &= 4 \end{aligned}$$

3.
$$\begin{aligned} 2u - 2v + 2w &= 6 \\ u + 2v + 2w &= 5 \\ 2u + v + w &= 4 \\ u - v - w &= -1 \end{aligned}$$

4.
$$\begin{aligned} 2a + 3b + 2c + d &= 4 \\ a + 2b + 3c + 2d &= 4 \\ 3a + 2b + c + d &= 4 \\ 2a + b + c + 3d &= 3 \\ a + b + 3c + 2d &= 3 \end{aligned}$$

5.
$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 + x_4 &= 4 \\ x_1 + 2x_2 + x_3 + 2x_4 &= 2 \\ 3x_1 + 2x_2 + 3x_3 + x_4 &= 6 \\ 2x_1 + x_2 + 2x_3 + 3x_4 &= 4 \\ x_1 + x_2 + x_3 - 2x_4 &= 2 \end{aligned}$$

6.
$$\begin{aligned} -u + 2v + w + 2x &= 5 \\ 2u + 3v - w - 2x &= -1 \\ 2u - 3v + 2w + x &= -1 \\ 3u - v - 2w + x &= -8 \\ u + v - 3w - x &= -6 \end{aligned}$$

15 Vector-space

15.1 Introduction

There are several areas in mathematics which seem to be completely different but still show many similarities.

Points on a plane ($\mathbb{R}^2$), in a volume ($\mathbb{R}^3$), vectors of real or complex numbers, polynomials, continuous functions, differentiable functions are some of many examples.

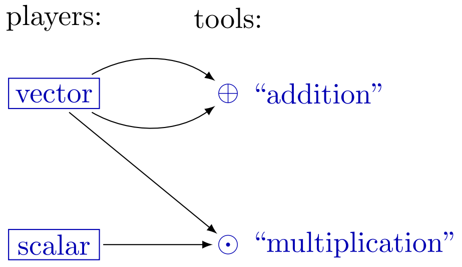
For all of them some sort of *addition* and *multiplication with scalar* are defined. Although being different fields of mathematics, many principles apply to all of them.

The concept of *vector-space* is an approach to raise the similarities. Once understood, they can be applied to many mathematical areas.

In this chapter we look at the general definition of vector-spaces and study its properties in the light of vectors, matrices and systems of linear equations.

15.2 Definition of vector-space

There are two players in a sandbox: a *vector* and a *scalar*. They play with two tools: "addition" of two vectors and the "multiplication" of a scalar and a vector.



Both tools, "addition" and "multiplication" result again in a vector.

Definition 15.1 (Field $\mathbb{K}$)**.** In the following $\mathbb{K}$ stands either for real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$ together with the known operators $+$ and $\cdot$ for addition and multiplication. ◁

Definition 15.2 (Vector space V over $\mathbb{K}$)**.** Let V be a set and $\mathbb{K}$ a field with the two operators $\oplus$ and $\odot$. For $u, v, w \in V$ and $\lambda, \mu \in \mathbb{K}$ let the following requirements be satisfied:

1. closure:

$$v \oplus w \in V$$
$$\lambda \odot v \in V$$

2. associativity:

$$(u \oplus v) \oplus w = u \oplus (v \oplus w)$$
$$\lambda \odot (\mu \odot v) = (\lambda \cdot \mu) \odot v$$

3. commutativity:

$$u \oplus v = v \oplus u$$

4. existence of neutral element:

$$v \oplus \vec{0} = v, \quad \vec{0} \in V$$
$$1 \odot v = v, \quad 1 \in \mathbb{K}$$

5. existence of inverse element:

$$v \oplus v' = \vec{0}, \quad v' \in V$$

6. distributivity:

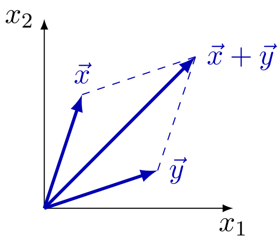
$$\lambda \odot (v \oplus w) = \lambda \odot v \oplus \lambda \odot w$$
$$(\lambda + \mu) \odot v = \lambda \odot v \oplus \mu \odot v$$

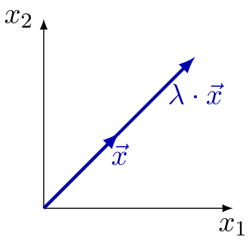
We then call V a *vector space* over $\mathbb{K}$. We call $v \in V$ a *vector* and $\lambda \in \mathbb{K}$ a *scalar*. ◁

Example 15.1. We define a vector-space for points on a plane: With $V = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, $c \in \mathbb{R}$ and $\vec{x}, \vec{y} \in V$, i.e. $\vec{x} = (x_1, x_2)^T$ and $\vec{y} = (y_1, y_2)^T$ for $x_1, x_2, y_1, y_2 \in \mathbb{R}$ we define:

- addition: $\vec{x} \oplus \vec{y} = (x_1 + y_1, x_2 + y_2)^T$
- multiplication: $c \odot \vec{x} = (c \cdot x_1, c \cdot x_2)^T$
- neutral element for $\oplus$: $\vec{0} = (0, 0)^T$
- inverse element for $\oplus$: $\vec{x}' = (-x_1, -x_2)^T$
- neutral element for $\odot$: 1

It is easy to verify that $(V, \oplus, \odot)$ is a vector space over $\mathbb{R}$ by checking against all axioms of vector-spaces. ◁





Example 15.2. Although not in our focus, here an example with different "addition" $\oplus$ and "multiplication" $\odot$: Let $V = \mathbb{R}_{>0} = (0, \infty)$ the set of all positive real numbers. With $x, y \in V$ and $\lambda \in \mathbb{R}$ we define:

- "addition": $x \oplus y = x \cdot y$
- "multiplication": $\lambda \odot x = x^\lambda$
- neutral element for $\oplus$: 1
- inverse element for $\oplus$: $x' = x^{-1} = \frac{1}{x}$
- neutral element for $\odot$: 1

By this definition V is a vector-space over $\mathbb{R}$. Verify it! ◁

Example 15.3. Let f and g be functions with domain $D \subseteq \mathbb{R}$ and:

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (\lambda f)(x) &= \lambda f(x)\end{aligned}$$

with $\lambda \in \mathbb{R}$. Some vector-spaces over $\mathbb{R}$:

- the set of all functions $f : D \to \mathbb{R}$
- the set of all functions $f : D \to \mathbb{C}$
- the set of all continuous functions
- the set of all differentiable functions
- the set of all polynomials

◁

Remark: Once $\oplus$ and $\odot$ have been defined it is common to use again $+$ and $\cdot$. For v' we write $-v$. Instead of writing $a+(-b)$ we write in short $a-b$.

Theorem 15.3 (Properties of vector-space)**.** Let V be a vector-space over $\mathbb{K}$, $v, \vec{0} \in V$ and $\lambda \in \mathbb{K}$.

- There exist exactly one zero-vector $\vec{0}$.
- For any v there exist exactly one inverse element $-v \in V$.
- $0 \odot v = \vec{0}$ and $(-1) \odot v = -v$
- $(-\lambda) \odot x = \lambda \odot (-x) = -(\lambda \odot x)$
- $(\lambda = 0) \vee (v = \vec{0}) \quad \leftrightarrow \quad \lambda \odot v = \vec{0}$

◁

Definition 15.4 (Column- and row-vector)**.** With $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{K}$ we define $\mathbb{K}^n$ as the set of all *column-vectors* noted as:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Similarly we define $\mathbb{K}^{n*}$ as the set of all *row-vectors* noted as:

$$x = (x_1, x_2, \dots, x_n)$$

◁

Theorem 15.5 (Column/row-vector-space)**.** A set of column-vectors with the element-wise addition

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix}$$

and the element-wise multiplication

$$\lambda \odot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}, \qquad \lambda \in \mathbb{K}$$

is a vector-space. The same applies to row-vectors $\mathbb{K}^{n*}$. ◁

Theorem 15.6 (Matrix-vector-space)**.** The set of matrices $M(m \times n, \mathbb{K})$ with

$$\begin{aligned}(a_{ij})_{m,n} \oplus (b_{ij})_{m,n} &= (a_{ij}+b_{ij})_{m,n}\\ \lambda \odot (a_{ij})_{m,n} &= (\lambda a_{ij})_{m,n}, \quad \lambda \in \mathbb{K}\end{aligned}$$

is a vector-space over $\mathbb{K}$. ◁

15.3 Subspace

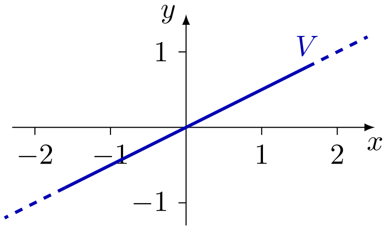
Example 15.4. Let V be defined by

$$V = \{(x, y) \mid 2y = x,\ \ x, y \in \mathbb{R}\}$$

with

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\\ \lambda \cdot (x, y) &= (\lambda x, \lambda y)\end{aligned}$$

The set V forms on a Cartesian diagram a straight line through the origin and fulfils all requirements for a vector-space. Hence, although being only a subset of $\mathbb{R}^2$, V itself is a vector space. ◁



Definition 15.7 (Subspace)**.** Let S be a subset of a vector-space V, i.e. $S \subseteq V$. If S is a vector-space itself, we call S a *subspace* of V. ◁

Theorem 15.8 (Conditions for a subspace)**.** Let V be a vector-space over $\mathbb{K}$ and S a subset of V, i.e. $S \subseteq V$. With $v, w \in S$ and $\lambda \in \mathbb{K}$ let the following be satisfied:

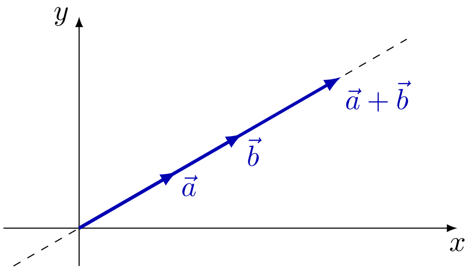
$$\begin{aligned}S &\neq \emptyset && \text{(non-empty set)}\\ v \oplus w &\in S && \text{(closure of } \oplus)\\ \lambda \odot v &\in S && \text{(closure of } \odot)\end{aligned}$$

Then S is a vector-space over $\mathbb{K}$. ◁

Remark: To verify whether a subset of a vector-space is a vector-space itself, not all axioms of definition 15.2 need to be tested. It is sufficient to check the two closure properties.

Example 15.5. For any vector-space V we have the so called *trivial-subspace* which contains only the neutral element 0 for $\oplus$: $S = \{0\}$. ◁

Example 15.6. For a two-dimensional real coordinate space $\mathbb{R}^2$ all coordinates on a straight line through the origin form a subspace. ◁



Example 15.7. For a three-dimensional real coordinate space $\mathbb{R}^3$ all coordinates on a flat surface through the origin form a subspace. ◁

Remark: We can not use our eyes to "see" real coordinate spaces with higher dimensions than three. However, from three-dimensional real coordinate spaces we may imagine a subspace as a flat surface through the origin.

15.4 Linear (in-) dependence

Definition 15.9 (Linear combination)**.** Let V be a vector space over $\mathbb{K}$, $v_1, \dots, v_n \in V$ and $\lambda_1, \dots, \lambda_n \in \mathbb{K}$. We call

$$\sum_{k=1}^{n} \lambda_k v_k = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n$$

a *linear combination* of $v_1, \dots, v_n$. ◁

Definition 15.10 (Linear span)**.** Let V be a vector-space over $\mathbb{K}$ and $v_1, \dots, v_n \in V$. We call the set of all possible linear combinations of $v_1, \dots, v_n$ *linear span* or just *span* of $v_1, \dots, v_n$:

$$\operatorname{span}(v_1, \dots, v_n) = \left\{ \sum_{k=1}^{n} \lambda_k v_k \,\middle|\, \lambda_k \in \mathbb{K} \right\}$$

If $\operatorname{span}(v_1, \dots, v_n) = V$ we call $\{v_1, \dots, v_n\}$ a *spanning set* of V. ◁

Example 15.8. Let V be the two-dimensional real coordinate space $\mathbb{R}^2$ with the standard operators $+$ and $\cdot$.

- The two unit vectors

  $$\vec{x} = \begin{pmatrix}1\\0\end{pmatrix} \qquad \text{and} \qquad \vec{y} = \begin{pmatrix}0\\1\end{pmatrix}$$

  form a spanning set for V.

- The two vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

  form a spanning set for V. I.e. any vector $\vec{v} = (x_1, x_2)^T \in V$ may be expressed by a linear combination of $\vec{v}_1$ and $\vec{v}_2$:

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= (x_1 - x_2)\vec{v}_1 + x_2\vec{v}_2 \\ &= (x_1 - x_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

- The two vectors

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \vec{u}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

  do not form a spanning set for V. E.g. it is impossible to express $(1, 0)^T$ by a linear combination of $\vec{u}_1$ and $\vec{u}_2$.

$\triangleleft$

Example 15.9.

- The unit-vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

  form a spanning-set for $\mathbb{R}^3$.

- The set of unit-vectors

$$\{e_1, e_2, \ldots, e_n\}$$

  forms a spanning-set for $\mathbb{R}^n$.

- The set

$$\{1, x, x^2, \ldots, x^n\}$$

  forms a spanning-set for all n^{th}-order polynomials.

$\triangleleft$

For a given set of vectors $\{v_1, v_2, \ldots, v_n\}$ there may or may not exist a dependency between the vectors. I.e. it may be possible to express one of the vectors by a linear combination of the other vectors of the set.

E.g. the third vector of the set

$$\{a_1, a_2, a_3\} = \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\}$$

can be expressed by the other vectors:

$$a_3 = 2a_1 - 3a_2$$

or in homogeneous form:

$$2a_1 - 3a_2 - a_3 = 0$$

Obviously, there is no dependency among the vectors of this set:

$$\{e_1, e_2, e_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

I.e. the homogeneous equation

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 = 0$$

holds only for the trivial case $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

This leads us to the following definition:

Definition 15.11 (Linear independence)**.** If for a set of vectors $\{v_1, \ldots, v_m\}$ there exist a set of scalars $\{\lambda_1, \ldots, \lambda_m\}$ with at least one of them not being zero where

$$\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_m v_m = 0$$

we say the set of vectors is *linear dependent.* If the above equation holds only for the trivial case, i.e. all scalars are zero, we say the set of vectors is *linear independent.* $\triangleleft$

Remark: To determine linear dependency we write the set of vectors as a matrix and change the matrix into reduced row echelon form. If the number of non-zero rows is less than the number of vectors, we have linear dependency.

Example 15.10. For the set of vectors in the previous example we get:

$$\begin{aligned} \operatorname{rref}((a_1, a_2, a_3)) &= \operatorname{rref}\begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

There are two non-zero rows, i.e. one less than the number of vectors. Hence, the set of vectors is linear dependent. $\triangleleft$

15.5 Basis and dimension

Definition 15.12 (Basis and dimension)**.** If a spanning set S of a vector-space V is linear independent we call S a *basis* of V. The number of vectors in the basis is the *dimension* of V and we write in short $\dim V$. ◁

Definition 15.13 (Standard basis)**.** For the vector-space $\mathbb{K}^n$ we call the set of unit vectors $\{e_1, e_2, \ldots, e_n\}$ the *standard basis* of $\mathbb{K}^n$. ◁

Example 15.11.

- The two vectors $v_1 = (1,1)^T$ and $v_2 = (-1,1)^T$ form a basis for the vector-space $V = \mathbb{R}^2$ where V has the dimension 2.
- The three vectors $v_1 = (1,0,0)$, $v_2 = (0,1,0)$ and $v_3 = (0,0,1)$ form the standard basis for the vector-space $V = \mathbb{R}^{3*}$ where V has the dimension 3.
- The set of unit-vectors $\{e_1, e_2, \ldots, e_n\}$ form the standard-basis for the vector space $V = \mathbb{R}^n$ where V has the dimension n.
- The set of vectors $\{v_1, v_2, v_3\}$ with $v_1 = (1,2,0)$, $v_2 = (2,3,-1)$ and $v_3 = (-1,0,2)$ do not form a basis for the vector-space $V = \mathbb{R}^{3*}$ since the set of vectors is linear dependent.

◁

Theorem 15.14 (Unique coefficients for basis vectors)**.** Any vector v of a vector-space V with basis $\{b_1, b_2, \ldots, b_n\}$ can be expressed by the basis and a set of unique scalars $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$:

$$v = \sum_{k=1}^{n} \lambda_k b_k = \lambda_1 b_1 + \lambda_2 b_2 + \ldots + \lambda_n b_n$$

◁

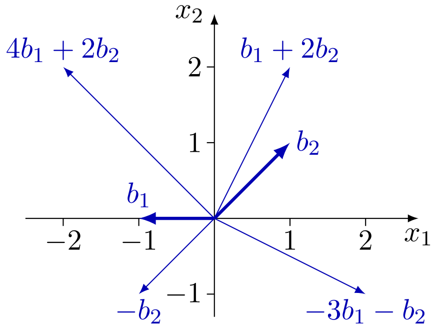
Example 15.12. Let $V = \mathbb{R}^2$ with basis $\{b_1, b_2\}$, $b_1 = (-1,0)^T$ and $b_2 = (1,1)^T$. Any vector $v = (v_1, v_2)^T \in V$ may be expressed by a linear combination of the basis with:

$$\begin{aligned} v = (v_1, v_2)^T &= (v_2 - v_1) b_1 + v_2 b_2 \\ &= (v_2 - v_1) \begin{pmatrix} -1 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

For example:

$$\begin{aligned} (-2,2)^T &= 4b_1 + 2b_2 \\ (1,2)^T &= b_1 + 2b_2 \\ (2,-1)^T &= -3b_1 - b_2 \\ (-1,-1)^T &= -b_2 \end{aligned}$$

◁



Remark: How do we find the factors $\lambda_1, \ldots, \lambda_n$ to express any vector $v \in V$ by the basis-vectors $b_1, \ldots, b_n$? We combine the basis-vectors as column-vector to a matrix $A = (a_{ij})_{m,n} = ((b_1, \ldots, b_n))$ and express the vector v by a product of matrix A with the column-vector of the factors $(\lambda_1, \ldots, \lambda_n)^T$:

$$\begin{aligned} v &= \lambda_1 b_1 + \lambda_2 b_2 + \ldots + \lambda_n b_n \\ &= \lambda_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \ldots + \lambda_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} \\ &= A \cdot \lambda \end{aligned}$$

Now we multiply the equation by the inverse of A and get:

$$A^{-1} \cdot v = A^{-1} \cdot A \cdot \lambda = I \cdot \lambda = \lambda$$

Hence, the inverse matrix A^{-1} multiplied with the vector v gives us the factors $\lambda_1, \ldots, \lambda_n$ for the base-vectors $b_1, \ldots, b_n$.

Example 15.13. Let $V = \mathbb{R}^3$ be a vector-space with the basis:

$$b_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \; b_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \; b_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

We combine the basis to a matrix and create the extended coefficient matrix

$$A = ((b_1, b_2, b_2)) = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

After applying elementary row operations:

$$(I \mid A^{-1}) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 2 & -1 & -2 \end{array}\right)$$

Now the factors $\lambda_1, \lambda_2, \lambda_3$ are

$$\begin{aligned} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} &= A^{-1} \cdot v \\ &= \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 2 \\ 2 & -1 & -2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\ &= \begin{pmatrix} -v_1 + v_2 + v_3 \\ -v_1 + 2v_3 \\ 2v_1 - v_2 - 2v_3 \end{pmatrix} \end{aligned}$$

I.e. any vector $v = (v_1, v_2, v_3)^T \in V$ may be expressed by the basis with:

$$\begin{aligned} v =& (-v_1 + v_2 + v_3)b_1 \\ &+ (-v_1 + 2v_3)b_2 \\ &+ (2v_1 - v_2 - 2v_3)b_3 \end{aligned}$$

E.g.

$$\begin{aligned} (1,1,1)^T &= b_1 + b_2 - b_3 \\ (1,2,3)^T &= 4b_1 + 5b_2 - 6b_3 \\ (3,2,1)^T &= -b_2 + 2b_3 \end{aligned}$$

$\triangleleft$

Remark: The basis of a subspace may have less vectors than elements in the vectors. In these situations the matrix of the basis-vectors has more rows than columns. We again apply the elementary row operations to the extended coefficient matrix and bring it into its reduced row echelon form to evaluate the factors for the basis-vectors.

Example 15.14. Let $S \subseteq \mathbb{R}^3$ be a subspace with basis:

$$b_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \; b_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

We get

$$\left(\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

and

$$\left(\begin{array}{cc|ccc} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right)$$

Since S is a subspace not all vectors $v \in \mathbb{R}^3$ are elements of S which becomes clear from the last row of the matrix. With $v = (v_1, v_2, v_3)^T \in S$ we have:

$$2v_1 - v_2 - 2v_3 = 0$$

and:

$$v = (v_2 - v_1)b_1 + (2v_1 - v_2)b_2$$

$\triangleleft$

15.6 Linear mapping

Yet we looked at vector-spaces itself only. In this section we will investigate linear functions that map one vector-space to another.

In previous chapters we looked already at functions that map one value to another. For example:

$$f : \begin{cases} \mathbb{R} \to \mathbb{C} \\ x \mapsto 2\mathrm{j}x \end{cases}$$

maps to every value in $\mathbb{R}$ a value in $\mathbb{C}$. In this section we want to map every vector of one vector-space to vectors of another (or the same) vector space.

However, out the large variety of functions we limit ourself to linear functions.

Definition 15.15 (Linear mapping)**.** Let V and W be vector-spaces over $\mathbb{K}$. A mapping $L : V \to W$ is called *linear* if

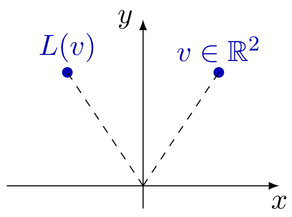
1. $L(x + y) = L(x) + L(y)$
2. $L(\lambda x) = \lambda L(x)$

for all $x, y \in V$ and $\lambda \in \mathbb{K}$. $\triangleleft$

Example 15.15. The linear map

$$L : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^2 \\ (x, y)^T \mapsto (-x, y)^T \end{cases}$$

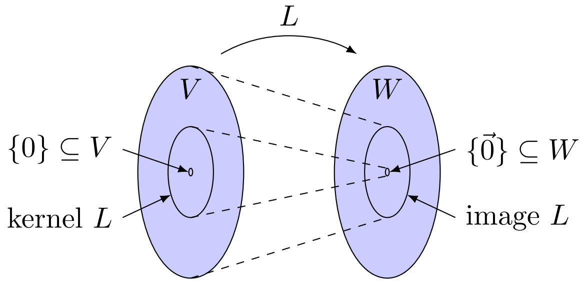
mirrors all points at the ordinate of a Cartesian coordinate system. $\triangleleft$



Definition 15.16 (Kernel and image)**.** Let V and W be vector-spaces over $\mathbb{K}$ and $L : V \to W$ a linear mapping. We define

$$\begin{aligned} \text{image } L &= L(V) = \{L(v) \mid v \in V\} \\ \text{kernel } L &= \{v \in V \mid L(v) = \vec{0}\} \subseteq V \end{aligned}$$

In words: The *image* of L is the set of all possible values of L. The *kernel* is the set of all $x \in X$ for which $L(x) = \vec{0}$. ◁



Example 15.16. For the linear map

$$L : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^2 \\ (x, y)^T \mapsto (x, 0)^T \end{cases}$$

we have

$$\begin{aligned} \text{image } L &= \{(x, 0)^T \mid x \in \mathbb{R}\} \\ \text{kernel } L &= \{(0, y)^T \mid y \in \mathbb{R}\} \end{aligned}$$

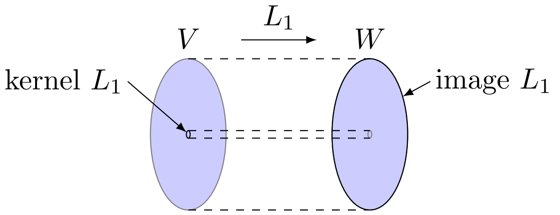
◁

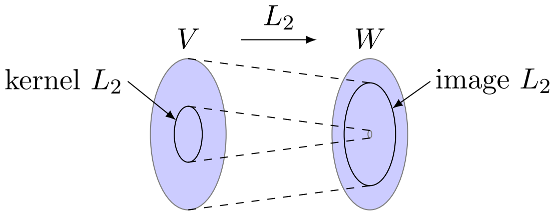
Theorem 15.17 (Kernel and image are subspaces)**.** Kernel and image of a linear mapping are subspaces, i.e. they fulfil all axioms of vector-spaces. ◁

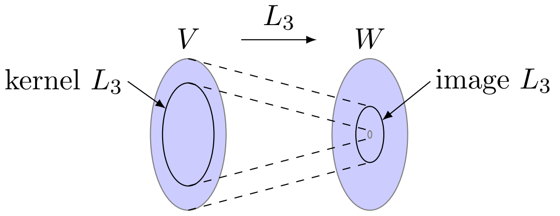
Theorem 15.18 (Rank-nullity theorem)**.** Let V and W be vector-spaces and $L : V \to W$ a linear mapping. We then have:

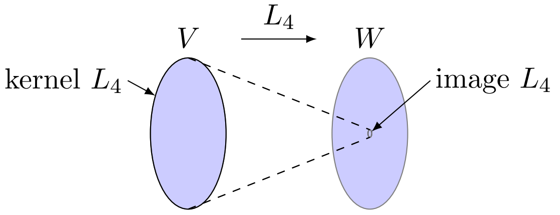
$$\dim\ V = \dim(\text{image } L) + \dim(\text{kernel } L)$$

◁









Example 15.17. For the linear map

$$L : \begin{cases} \mathbb{R}^{3*} \to \mathbb{R}^{2*} \\ (x, y, z) \mapsto (x - y, y - x) \end{cases}$$

we have

$$\begin{aligned} \text{image } L &= \{(x, -x) \mid x \in \mathbb{R}\} \\ \text{kernel } L &= \{(x, -x, y) \mid x, y \in \mathbb{R}\} \end{aligned}$$

The image of L has a dimension of 1 and the kernel of L has a dimension of 2. The sum equals the dimension of the domain which is three. ◁

15.7 Matrices as linear mapping

Theorem 15.19 (Unique coefficients)**.** Let X be a vector-space over $\mathbb{K}$ with base $B_X = \{x_1, \ldots, x_n\}$ and Y be a vector-space over $\mathbb{K}$ with base $B_Y = \{y_1, \ldots, y_m\}$. Let further $L : X \to Y$ be a linear mapping.

We then have for every base-vector $x_j \in B_X$ unique scalars $a_{1j}, \ldots, a_{mj}$ such that

$$L(x_j) = a_{1j}y_1 + a_{2j}y_2 + \ldots + a_{mj}y_m$$

◁

Definition 15.20 (Transformation matrix)**.** We call the matrix constructed by the elements a_{ij} of theorem 15.19 *transformation matrix*. ◁

Example 15.18. We want to express the linear map

$$L: \begin{cases} \mathbb{R}^3 \to \mathbb{R}^3 \\ (x,y,z)^T \mapsto (2x-y, x+z, y-2z)^T \end{cases}$$

by a transformation matrix. With $v, v' \in \mathbb{R}^3$ we get:

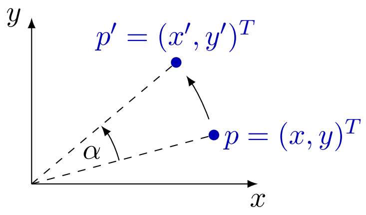
$$\begin{array}{ccccc} A & \cdot & v & = & v' \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} & \cdot & \begin{pmatrix} x \\ y \\ z \end{pmatrix} & = & \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \end{array}$$

Comparing with $L(v) = v'$ we get:

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

◁

Example 15.19. A transformation matrix may be used to rotate points around the origin of a Cartesian coordinate system.



Empirically we derive the equations:

$$x' = x\cos\alpha - y\sin\alpha$$
$$y' = x\sin\alpha + y\cos\alpha$$

which we may write in vector/matrix notation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$
$$p' = A \cdot p$$

◁

Theorem 15.21 (Rank and dimension of image)**.** The rank of a transformation matrix equals the dimension of its image, i.e. with $A = M(m \times n, \mathbb{K})$ we have:

$$\mathrm{rank} A = \dim(\mathrm{image}\, A)$$

◁

Remark: The evaluation of the rank of a transformation matrix gives the dimension of its image. This may be done by changing it into its reduces row echelon form.

15.8 System of linear equations

A system of linear equations, SLE, may be looked at as a vector-space for the unknowns which is linearly mapped by a matrix to another vector-space for the constants.

Example 15.20. For an SLE with 4 equations and 3 unknowns on real numbers we have:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 &= b_4 \end{aligned}$$

or

$$\begin{array}{ccccc} A & \cdot & x & = & b \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} & \cdot & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & = & \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \end{array}$$

with

$$\begin{aligned} x &= (x_1, x_2, x_3)^T \in \mathbb{R}^3 \\ b &= (b_1, b_2, b_3, b_4)^T \in \mathbb{R}^4 \\ A &= (a_{ij}) = M(4 \times 3, \mathbb{R}) \end{aligned}$$

The matrix A acts like a linear function:

$$A: \begin{cases} \mathbb{R}^3 \to \mathbb{R}^4 \\ x \mapsto b = (a_{ij})_{4,3} \cdot x \end{cases}$$

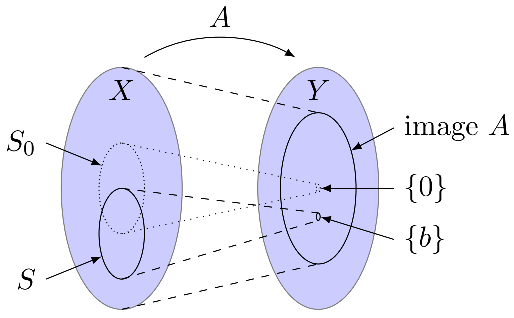
◁

Theorem 15.22 (Solution set of an SLE)**.** Let $A = M(m \times n, \mathbb{K})$ and $b = \mathbb{K}^m$. Let further be $x_1 \in \mathbb{K}^n$ be a particular solution of the equation $Ax_1 = b$ and S_0 be the set of solutions of the homogeneous equation $Ax = 0$.

We then have for the solution set S of $Ax = b$:

$$S = \{x_1 + x_0 \mid x_0 \in S_0\}$$

◁



We saw earlier that an SLE has three solutions behaviours: no solution, single solution and infinite number of solutions. With x as the vector of unknowns, A as the linear mapping matrix and b as the vector of constants we find these behaviours again:

- If the constant vector b is not an element of the image of the matrix A we have no solution, i.e.

$$b \notin \text{image } A \quad \leftrightarrow \quad \text{no solution}$$

- If b is an element of the image of A and the dimension of the kernel of A is zero we have a single solution, i.e.

$$\begin{aligned} b \in \text{image}\, A \ \wedge \ \dim(\text{kernel}\, A) = 0 \\ \leftrightarrow \quad \text{single solution} \end{aligned}$$

- If b is an element of the image of A and the dimension of the kernel of A is greater than zero we have an infinite number of solutions, i.e.

$$\begin{aligned} b \in \text{image}\, A \ \wedge \ \dim(\text{kernel}\, A) > 0 \\ \leftrightarrow \quad \text{infinite number of solutions} \end{aligned}$$

How do we find out, whether $b \in \text{image}\, A$? We change the extended coefficient matrix $(A \mid b)$ into its reduced row echelon form. If rank(A) equals rank$(A \mid b)$ we know that b is an element of image(A). Otherwise b is no element of image(A).

To derive dim(kernel A) we need the following theorem:

Theorem 15.23. With $A \in M(m \times n, \mathbb{K})$ we have:

$$\dim(\text{kernel}\, A) = n - \text{rank } A$$

◁

Example 15.21. The SLE

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 3x_3 & = & 2 \\ 2x_1 & + & x_2 & + & 2x_3 & = & 2 \\ 3x_1 & + & x_2 & + & x_3 & = & 2 \end{array}$$

has the extended coefficient matrix:

$$(A \mid b) = \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 2 & 1 & 2 & 2 \\ 3 & 1 & 1 & 2 \end{array}\right)$$

and in reduced row echelon form:

$$\left(\begin{array}{ccr|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

I.e. we have

$$\begin{aligned} &b \in \text{image}\, A \\ &\text{rank}\, A = 2 \\ &\dim(\text{kernel}\, A) = 1 \end{aligned}$$

Hence, the SLE is solvable with an infinite number of solutions. ◁

15.9 Problems

Problem 15.1: Let $V = \mathbb{R}^{2*}$ be a set of row-vectors with element-wise addition and multiplication. Show that V is a vector space by checking all axioms.

Problem 15.2: Let $V = M(2\times 2, \mathbb{R})$ be the set of all real-valued 2×2 matrices. With element-wise addition and multiplication, show that V is a vector-space.

Problem 15.3: Is the set

$$V = \{(x, y)^T \in \mathbb{R}^2 \mid x = 2t, y = 3t, t \in \mathbb{R}\}$$

with element-wise addition and multiplication a vector-space?

Problem 15.4: Which of the following sets are subsets of $\mathbb{R}^{2*}$ with element-wise addition and multiplication?

1. $\{(x, y) \mid y = 2x\}$
2. $\{(x, y) \mid y = \sqrt{x}\}$
3. $\{(x, y) \mid x = 0\}$
4. $\{(x, y) \mid x = 1\}$
5. $\{(x, y) \mid y \geq 0\}$
6. $\{(x, y) \mid xy = 0\}$

Problem 15.5: Let V be the set of all points on an infinite sized plane passing through the origin of a three dimensional coordinate system $\mathbb{R}^3$. Let W be the set of all points on a straight line through the origin perpendicular to the plane of V.

1. Is V a vector-space?
2. Is W a vector space?

3. Is the set $\{v + w \mid v \in V, w \in W\}$ a vector space?

Problem 15.6: Derive the vector-spaces spanned by the following spanning-sets:

1. $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ 2. $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\right\}$
3. $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\}$ 4. $\{e_1, e_2, e_3, e_4\}$
5. $\left\{\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}\right\}$

Problem 15.7: Which of the following vector-sets are linearly independent?

1. $\left\{\begin{pmatrix}3\\4\\5\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\3\\3\end{pmatrix}\right\}$
2. $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}3\\2\\3\end{pmatrix}, \begin{pmatrix}6\\6\\8\end{pmatrix}\right\}$
3. $\left\{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}3\\3\\2\\1\end{pmatrix}, \begin{pmatrix}4\\1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\1\\2\end{pmatrix}\right\}$
4. $\left\{\begin{pmatrix}2\\0\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\1\\2\\2\end{pmatrix}, \begin{pmatrix}3\\0\\2\\2\end{pmatrix}\right\}$

Problem 15.8: Give the standard basis of the following vector spaces:

1. $\mathbb{R}^2$ 2. $\mathbb{R}^3$ 3. $\mathbb{R}^n$

Problem 15.9: Find three different basis for each of the following vector-spaces:

1. $\mathbb{R}^2$ 2. $\mathbb{R}^3$ 3. $\{(x, y, z) \in \mathbb{R}^{3*} \mid x = y\}$

Problem 15.10: Let $V = \mathbb{R}^{3*}$ be a vector-space with basis $\{b_1, b_2, b_3\}$, $b_1 = (1, 1, 2)$, $b_2 = (1, 2, 0)$ and $b_3 = (2, 3, 1)$. Express the following vectors by this basis.

$$v_1 = (1, 1, 1) \qquad v_3 = (-2, 0, 2)$$
$$v_2 = (1, 2, 3) \qquad v_4 = (-3, -2, -1)$$

Problem 15.11: Let $S \subseteq \mathbb{R}^3$ be a subspace with basis $\{b_1, b_2\}$, $b_1 = (1, 2, 3)^T$, $b_2 = (3, 2, 1)^T$. Try to express the following vectors by this basis.

$$v_1 = (1, 1, 1)^T \qquad v_3 = (2, 1, 2)^T$$
$$v_2 = (-1, 1, 3)^T \qquad v_4 = (6, 2, -2)^T$$

Problem 15.12: Which of the following mappings for vector-spaces with element-wise addition and multiplication are linear?

$$L_1 : \mathbb{R}^2 \to \mathbb{R}^2, (x, y)^T \mapsto (x, y)^T$$
$$L_2 : \mathbb{R}^{2*} \to \mathbb{R}^{2*}, (x, y) \mapsto (2y, 3x)$$
$$L_3 : \mathbb{R}^2 \to \mathbb{R}^{2*}, (x, y)^T \mapsto (xy, yx)$$
$$L_4 : \mathbb{R}^2 \to \mathbb{R}^2, (x, y)^T \mapsto (x^2, \sqrt{y})^T$$
$$L_5 : \mathbb{R}^2 \to \mathbb{R}^3, (x, y)^T \mapsto (x, y, x + y)^T$$
$$L_6 : \mathbb{R}^{3*} \to \mathbb{R}^{2*}, (x, y, z) \mapsto (x + y, y + z)$$

Problem 15.13: Derive the transformation matrices of the following linear maps.

$$L_1 : \mathbb{R}^2 \to \mathbb{R}^2, (x, y)^T \mapsto (2x, y/2)^T$$
$$L_2 : \mathbb{R}^2 \to \mathbb{R}^2, (x, y)^T \mapsto (0, 0)^T$$
$$L_3 : \mathbb{R}^2 \to \mathbb{R}^3, (x, y)^T \mapsto (x, y, 0)^T$$
$$L_4 : \mathbb{R}^3 \to \mathbb{R}^2, (x, y, z)^T \mapsto (x + y, y + z)^T$$

Problem 15.14: For the linear maps of the previous problem find the dimensions of domain, image and kernel.

Problem 15.15: What is the effect of the following transformation matrices that map $\mathbb{R}^2$ to $\mathbb{R}^2$, $k, \alpha \in \mathbb{R}$?

1. $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ 3. $\begin{pmatrix}\cos\alpha & \sin\alpha\\-\sin\alpha & \cos\alpha\end{pmatrix}$
2. $\begin{pmatrix}k & 0\\0 & k\end{pmatrix}$ 4. $\begin{pmatrix}k\cos\alpha & -k\sin\alpha\\k\sin\alpha & k\cos\alpha\end{pmatrix}$

Problem 15.16: For problem 14.10 of the previous chapter evaluate the dimensions of the domain, image and kernel of the coefficient matrix.

16 Determinants

16.1 Introduction

As you may have learned in school, determinants have something to do with matrices. However, the concept of determinants are older than the concept of matrices!

The word *determinant* comes from *determine* and it is about *to determine* whether an system of linear equations (SLE) is solvable. It was decades later that an SLE was formalized by matrices.

The determinant is also useful to determine the surface on a parallelogram or the volume of a parallelepiped. Furthermore, determinants may be used to actually solve an SLE. These are the topics we want to handle in this chapter.

16.2 Determinant of 2×2 matrix

We look at a system of linear equations (SLE) with two equations and two unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

Under which conditions is this SLE solvable with a single solution? We solve the SLE with respect to x_1 and x_2 and get:

$$\begin{aligned} x_1 &= \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} \\ x_2 &= \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}} \end{aligned}$$

The denominator of both equations are equal and must be non-zero to derive a single solution for x_1 and x_2, i.e.

$$a_{11}a_{22} - a_{12}a_{21} \neq 0 \quad \text{(cond. for single sol.)}$$

To determine whether the SLE is uniquely solvable we evaluate the so called determinant:

Definition 16.1 (Determinant of 2×2 matrix)**.** With $A = (a_{ij}) \in M(2 \times 2, \mathbb{K})$ we call

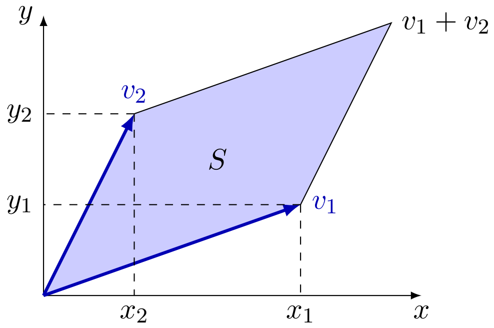
$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \in \mathbb{K}$$

the *determinant* of A. In short we write:

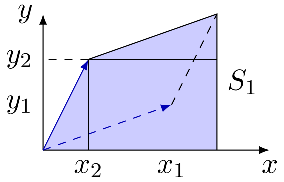
$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

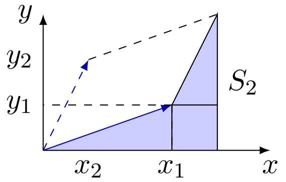
◁

The determinant may be interpreted as the surface S of a parallelogram spanned by the two column-vectors $v_1 = (x_1, y_1)^T$ and $v_2 = (x_2, y_2)^T$.



We separate the parallelogram into an area S_1 below the two upper lines and an area S_2 below the two lower lines.





$$\begin{aligned} S_1 &= \frac{x_2y_2}{2} + x_1y_2 + \frac{x_1y_1}{2} \\ S_2 &= \frac{x_1y_1}{2} + x_2y_1 + \frac{x_2y_2}{2} \\ S &= S_1 - S_2 \\ &= \frac{x_2y_2}{2} + x_1y_2 + \frac{x_1y_1}{2} \\ &\quad - \frac{x_1y_1}{2} - x_2y_1 - \frac{x_2y_2}{2} \\ &= x_1y_2 - x_2y_1 \end{aligned}$$

Using the standard notation of a matrix

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A$$

we get

$$S = a_{11}a_{22} - a_{12}a_{21}$$

which looks much like the determinant of A. However, the determinant of A may take negative values. Hence we write:

$$S = |\det(A)|$$

From this graphical representation of the determinant of a 2×2 matrix A we can derive the following properties:

- If the two column-vectors of A are equal, the determinant of A is zero.
- If the two column-vectors of A point exactly in the same or opposite direction, the determinant of A is zero.
- If one of the column-vectors of A is zero, the determinant of A is zero.

16.3 Determinant of 3×3 matrix

We look at an SLE with three equations and three unknowns.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

When solving towards x_1, x_2 and x_3 we get some fairly long equations. However, the tree equations all have the same denominator. If this denominator is non-zero the SLE is solvable with a single solution:

$$\begin{aligned} &a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \neq 0 \\ &\qquad \text{(condition for single solution)} \end{aligned}$$

To determine whether the SLE has a single solution we define the determinant:

Definition 16.2 (Determinant of 3×3 matrix)**.** With $A = (a_{ij}) \in M(3 \times 3, \mathbb{K})$ we call

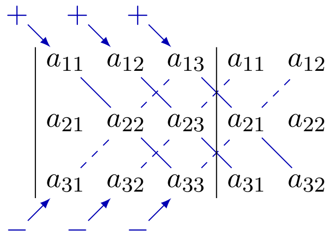
$$\begin{aligned} \det(A) = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \end{aligned}$$

the *determinant* of A. In short we write:

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

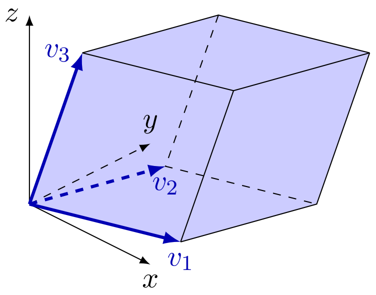
◁

To memorize the equation we write down the 3×3 matrix and repeat column 1 and 2 once more to the right of the matrix. Now we add the products of the three diagonals pointing towards south-east and subtract the products of the three diagonals pointing towards north-east. We call this the *rule of Sarrus*.



The absolute determinant of a 3×3 matrix A gives the volume V of a parallelepiped if its spanning vectors v_1, v_2 and v_3 are combined as row- or column-vectors into a matrix, i.e.

$$V = |\det(A)|$$



We observe:

- if all vectors are equal the determinant is zero
- if all vectors are on a line or on a plane the determinant is zero
- if one vector is zero the determinant is zero

Transferred to a matrix we observe: if the columns or rows are linear dependent, the determinant is zero.

16.4 Determinant of n×n matrix

How do we evaluate the determinant for a square matrix with arbitrary size? Unfortunately, the rule of Sarrus can not be applied for matrices larger than 3×3. Hence, we need a general method to evaluate the determinant of a square matrix. To get there we need the term *minor* of a matrix A:

Definition 16.3 (Minor of a matrix)**.** The *minor* $M_{ij} \in M(n-1 \times n-1, \mathbb{K})$ of a matrix $A \in M(n \times n, \mathbb{K})$ is the determinant of matrix A where row i and column j have been removed. ◁

Example 16.1. The matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

has the minors

$$\begin{aligned} M_{11} &= \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}, & M_{12} &= \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}, & M_{13} &= \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}, \\ M_{21} &= \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}, & M_{22} &= \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}, & M_{23} &= \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}, \\ M_{31} &= \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}, & M_{32} &= \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}, & M_{33} &= \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}, \end{aligned}$$

i.e.

$$\begin{aligned} M_{11} &= -3, & M_{12} &= -6, & M_{13} &= -3, \\ M_{21} &= -6, & M_{22} &= -12, & M_{23} &= -6, \\ M_{31} &= -3, & M_{32} &= -6, & M_{33} &= -3. \end{aligned}$$

◁

Definition 16.4 (Laplace expansion)**.** Let $A = (a_{ij})_{n,n}$ and M_{ij} its minors. We define the determinant by

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

for any $i \in \{1, \dots, n\}$ (column expansion) or

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

for any $j \in \{1, \dots, n\}$ (row expansion). ◁

Example 16.2. We want to evaluate the determinant of the matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

We use Laplace expansion on the first row:

$$\begin{aligned} \det(A) &= +a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \\ &= +1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= 1 \cdot (-3) - 2 \cdot (-6) + 3 \cdot (-3) = 0 \end{aligned}$$

or on the second column:

$$\begin{aligned} \det(A) &= -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32} \\ &= -2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5 \cdot \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8 \cdot \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \\ &= -2 \cdot (-6) + 5 \cdot (-12) - 8 \cdot (-6) = 0 \end{aligned}$$

◁

Remark: When the determinant of a larger matrix is evaluated it is useful to expand over rows or column with many zeros.

Example 16.3. We want to evaluate the determinant of the matrix:

$$A = \begin{pmatrix} 2 & 4 & 3 & 0 \\ 0 & 5 & 3 & 0 \\ 3 & 0 & 4 & 2 \\ 0 & 3 & 5 & 0 \end{pmatrix}$$

We chose the last column to apply Laplace expansion:

$$\begin{pmatrix} 2 & 4 & 3 & 0 \\ 0 & 5 & 3 & 0 \\ 3 & 0 & 4 & \boxed{2} \\ 0 & 3 & 5 & 0 \end{pmatrix}$$

The 2 is in the third row ($i = 3$) and fourth column ($j = 4$), i.e. $(-1)^{3+4} = -1$. We get:

$$\det(A) = -2 \cdot \begin{vmatrix} 2 & 4 & 3 \\ 0 & 5 & 3 \\ 0 & 3 & 5 \end{vmatrix}$$

Then we choose the first column to apply once more Laplace expansion:

$$\begin{pmatrix} \boxed{2} & 4 & 3 \\ 0 & 5 & 3 \\ 0 & 3 & 5 \end{pmatrix}$$

For the 2 in the first row and column we get a positive sign, hence:

$$\begin{aligned} \det(A) &= -2 \cdot 2 \cdot \begin{vmatrix} 5 & 3 \\ 3 & 5 \end{vmatrix} \\ &= -2 \cdot 2 \cdot (5 \cdot 5 - 3 \cdot 3) = -64 \end{aligned}$$

◁

Although we can not *see* more than three dimensions we generally may look at the absolute of determinants as the volume of parallelepipeds spanned by the row- or column-vectors of the matrices. This gives us a better understanding for some of the following properties.

16.5 Properties of determinants

Theorem 16.5 (Properties of determinants)**.** Let $A \in M(n \times n, \mathbb{K}$ be a square matrix.

- If all rows or all columns of a A are equal: $\det(A) = 0$
- If one row or column of matrix A is zero: $\det(A) = 0$
- If the rows or columns of A are linearly dependent: $\det(A) = 0$

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Theorem 16.6 (Determinant and invertible matrix)**.** A matrix A is invertible if and only if its determinant is not zero:

$$\det A \neq 0 \quad \leftrightarrow \quad A \text{ invertible}$$

◁

Theorem 16.7 (Determinants and elementary row operations)**.** Let $A = M(n \times n, \mathbb{K})$ be a square matrix. We then have for

row switching: With A' being matrix A with two rows exchanged we get:

$$\det(A') = -\det(A)$$

row multiplication: With A' being matrix A with one row multiplied by $\lambda \in \mathbb{K}$ we get:

$$\det(A') = \lambda \det(A)$$

row addition: With A' being matrix A with a multiple of one row added to another row we get:

$$\det(A') = \det(A)$$

◁

Theorem 16.8 (Further properties of determinants)**.** Let $A, B = M(n \times n, \mathbb{K})$ be square matrices and $\lambda \in \mathbb{K}$. We then have

- $\det A = \det A^T$
- $\det(\lambda A) = \lambda^n \det A$
- $\det(A \cdot B) = \det A \cdot \det B$
- $\det A^{-1} = (\det A)^{-1} \quad \text{for } \det(A) \neq 0$

◁

Theorem 16.9 (Determinant of special matrices)**.**

triangular matrix. The determinant of a triangular matrix is the product of its diagonal elements:

$$\det(A_{\text{triangle}}) = \prod_{k=1}^{n} a_{11} = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$$

diagonal matrix. The determinant of a diagonal matrix is the product of its diagonal elements:

$$\det(A_{\text{diagonal}}) = \prod_{k=1}^{n} a_{11} = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$$

identity matrix. The determinant of an identity matrix is one:

$$\det(I_n) = 1$$

◁

Example 16.4. The determinant of

$$A = \frac{1}{2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \quad \text{is}$$

$$\det(A) = \frac{1}{2^3} \cdot 1 \cdot 4 \cdot 6 = 3$$

◁

Example 16.5. For the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix}$$

we want to derive the determinant of their product $\det(A \cdot B)$.

We discover linear dependency for the rows of B (row 3 is the sum of the other two), hence:

$$\det(A \cdot B) = \det(A) \cdot \det(B) = \det(A) \cdot 0 = 0$$

◁

Example 16.6. We were able to convert a matrix A by elementary row operations into an identity matrix. We did not exchange any rows, multiplied rows by the factors $\lambda_1 = 2$, $\lambda_2 = \frac{1}{2}$ and $\lambda_3 = \frac{1}{3}$ and performed some row additions. Are we able to derive $\det(A)$?

We investigate the effect of row-operations on the determinant and find:

$$\begin{aligned} \det(I_n) &= \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \det(A) \\ \det(A) &= \frac{\det(I_n)}{\lambda_1 \cdot \lambda_2 \cdot \lambda_3} = \frac{1}{2 \cdot \frac{1}{2} \cdot \frac{1}{3}} = 3 \end{aligned}$$

◁

16.6 Cramer's rule

Determinants are an useful tool to determine whether an SLE with n equations and n unknowns is solvable. But can we use determinants to actually solve the SLE?

First we look at an SLE with two equations and two unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$A \cdot x = b$$

with the solution:

$$\begin{aligned} x_1 &= \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \\ x_2 &= \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}} \end{aligned}$$

In the denominator we find $\det(A)$. The numerator equals the determinant of a modified matrix A_i:

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

For A_i we replaced column j by the constant vector b.

The same is possible for larger matrices:

Theorem 16.10 (Cramer's rule)**.** We consider a system of linear equations

$$A \cdot x = b$$

with $A \in M(n \times n, \mathbb{K})$ as the coefficient matrix, $x = (x_1, \dots, x_n)^T \in \mathbb{K}^n$ as the vector of unknowns and $b \in \mathbb{K}^n$ as the vector of constants. We then get

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i denotes the matrix A where column i is replaced by the vector of constants b. ◁

Example 16.7. We consider the SLE

$$\begin{aligned} 3x_1 + 2x_2 &= 9 \\ 2x_1 + 3x_2 &= 1 \end{aligned}$$

and get

$$\begin{array}{ccccc} A & \cdot & x & = & b \\ \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} & \cdot & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & = & \begin{pmatrix} 9 \\ 1 \end{pmatrix} \end{array}$$

Applying Cramer's rule we derive:

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 9 & 2 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}} = \frac{9 \cdot 3 - 1 \cdot 2}{3 \cdot 3 - 2 \cdot 2} = 5$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 3 & 9 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}} = \frac{3 \cdot 1 - 2 \cdot 9}{3 \cdot 3 - 2 \cdot 2} = -3$$

◁

Although Cramer's rule is a simple method to solve an SLE, it is very laborious to perform. For an SLE with n equations and n unknowns we must evaluate $n + 1$ determinants of $n \times n$ matrices.

Hence, in most situations it is more efficient to bring the extended coefficient matrix into its reduced row echelon form and then to derive the unknowns.

16.7 Problems

Problem 16.1: Derive the determinant of the following matrices.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Problem 16.2: Determine the area of the parallelograms spanned by the following vectors.

1. $v_1 = (1, 2)$ and $v_2 = (-1, 3)$
2. $v_1 = (-3, 6)$ and $v_2 = (2, -4)$
3. $v_1 = (4, 5)$ and $v_2 = (5, 4)$

Problem 16.3: Derive the determinant of the following matrices.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 2 \\ 3 & 6 & 2 \end{pmatrix} \qquad F = \begin{pmatrix} 5 & 2 & 8 \\ 3 & 4 & 1 \\ 7 & 2 & 6 \end{pmatrix}$$

Problem 16.4: Determine the volume of the parallelepiped spanned by the vectors $v_1 = (1, 2, 3)$, $v_2 = (2, 2, -1)$ and $v_3 = (-1, 3, 3)$.

Problem 16.5: Derive the determinant of the following matrices.

$$A = \begin{pmatrix} 0 & 3 & 3 & 4 \\ 2 & 3 & 4 & 2 \\ 4 & 0 & 0 & 2 \\ 1 & 3 & 5 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 4 & 4 \\ 0 & 5 & 3 & 0 \\ 1 & 1 & 2 & 0 \\ 5 & 4 & 3 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 5 & 1 & 5 \\ 2 & 1 & 1 & 2 \\ 4 & 1 & 0 & 4 \\ 2 & 3 & 0 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 5 & 1 & 0 & 3 \\ 1 & 0 & 3 & 2 & 0 \\ 2 & 0 & 4 & 1 & 2 \\ 2 & 0 & 1 & 3 & 0 \\ 1 & 5 & 4 & 5 & 2 \end{pmatrix}$$

Problem 16.6: We applied elementary row operations on a matrix A and converted it into an identity matrix. We multiplied rows by the factors $\lambda_1 = 6$, $\lambda_2 = \frac{1}{8}$, $\lambda_3 = \frac{1}{3}$ and $\lambda_4 = 2$. We exchanged row 1 with row 3 and added multiples of rows to other rows. What is the determinant of A?

Problem 16.7: Let $A, B \in M(4 \times 4, \mathbb{K})$ with $\det(A) = 6$ and $\det(B) = \frac{1}{4}$. Solve the following expressions.

1. $\det(A^T)$
2. $\det(B^T)$
3. $\det(A^{-1})$
4. $\det(B^{-1})$
5. $\det(2 \cdot (A^{-1}))$
6. $\det(2 \cdot (B^{-1}))$
7. $\det((2 \cdot A)^{-1})$
8. $\det((2 \cdot B)^{-1})$
9. $\det(A \cdot B)$
10. $\det(B^T \cdot A)$
11. $\det(A \cdot A^{-1})$
12. $\det(A \cdot B^{-1})$

Problem 16.8: Solve the first three SLEs of problem 13.1 by Cramer's rule.

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